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The Statistics of Honeycomb and Triangular Lattice. I.

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Summary

After the matrix method in crystal statistics was introduced by R. Kubo[1] Kramers and Wannier[2], and others, Onsager[3], using the abstract algebraic method, got the exact solution in the case of plane square net. His rather complicated method was simplified by Nambu[6] considerably. We, independenty, attained the simplification of Onsager's method and have applied to the honeycomb lattice. And the exact solution has been obtained. From this, by the so called dual transformation we can get the partition function for triangular net.

The Curie point occurs at ch $2H = 2$ in the honeycomb lattice and at $\exp(4H) = 3$ in the triangular net. These coincide with the results obtained by the dual and star-triangle transformations[4]. The specific heat becomes logarithmically infinite but the energy itself remains continuous at this temperature. In the antiferromagnetic case, the honeycomb lattice behaves similar to the former case but the triangular net exhibits no phase change.

1. Eigenvalue Operator

In applying the matrix method, we take as one tier the vertical set of points $(1, 2, 3 \dots n)$ and the next tier $(1', 2', \dots n')$. (See fig. 1) To avoid the edge effect, the n-th atoms are considered to be connected to the first atoms. Then in this case n must be even. As the most simple case, we use the Ising model. Namely each lattice point is occupied by the atoms having $(+)$ or $(-)$ spin and the interaction energy of the nearest neighbor is $J/2$ or $-J/2$ according as the neighboring spins are antiparallel or parallel. The interactions except the nearest are neglected. By μ_i, which takes the value 1 or -1, we designate the spin on the i-th atom in the tier. Then putting matrix V as follows

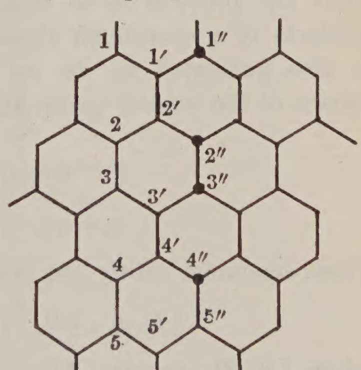

Fig. 1.

$$V(\mu_1, \mu_2, \cdots \mu_n;\ \mu_1'', \mu_2'' \cdots \mu_n'') = \sum_{<\mu'>} \exp H(\mu_2\mu_3 + \mu_4\mu_5 + \cdots) \exp H(\mu_1\mu_1' + \mu_2\mu_2' + \cdots)$$

$$\times \exp H(\mu_1'\mu_2' + \mu_3'\mu_4' + \cdots) \exp H(\mu_1'\mu_1'' + \mu_2'\mu_2'' + \cdots)$$

where $H=J/2kT$. The eigenvalue problem becomes

$$\sum_{<\mu''>} V(\mu_1, \mu_2, \cdots \mu_n;\ \mu_1'', \mu_2'' \cdots \mu_n'')\ \varphi(\mu_1'', \cdots \mu_n'') = \lambda\varphi(\mu_1, \cdots \mu_n).$$

The partition function becomes by their eigenvalues $\lambda_1, \lambda_2, \cdots$

$$f = \lambda_1^N + \lambda_2^N + \cdots$$

where N is the number of tiers divided by 2. By the theorem of Frobenius the maximum eigenvalue of a matrix, whose elements are all positive, is positive, simple and the eigenfunction is even. If λ_1 is assumed to be maximum, then as $N \to \infty$, $f=\lambda_1^N$. So that to get the partition function, we must solve the eigenvalue problem of V. As in Onsager's case, we introduce the operators c_i and s_i as follows

$$c_i f(\mu_1, \cdots, \mu_i, \cdots \mu_n) = f(\mu_i, \cdots, (-)\mu_i, \cdots \mu_n),$$

$$s_i f(\mu_1, \cdots, \mu_i, \cdots \mu_n) = \mu_i f(\mu, \cdots, \mu_i, \cdots \mu_n).$$

Then it is evident that

$$c_i^2 = s_i^2 = 1,\ c_i s_i = -s_i c_i,\ c_i c_k = c_k c_i,\ s_i s_k = s_k s_i,\ c_i s_k = s_k c_i\ \ (i \neq k)$$

By using these, V can be written as

$$V = (2\,\mathrm{sh}\,2H)^n \exp H(s_2 s_3 + s_4 s_5 + \cdots) \exp H^*(c_1 + c_2 + \cdots + c_n)$$
$$\times \exp H(s_1 s_2 + s_3 s_4 + \cdots) \exp H^*(c_1 + c_2 + \cdots + c_n).$$

Here H^* is connected with H as follows

$$e^{-2H^*} = \tanh \cdot H.$$

The relation between H^* and H is symmetrical.

Thus the problem is to diagonalise the operator V in the space of the direct products of n quaternion algebras $(1, c_i, s_i, c_i s_i)\ (i=1, 2, 3 \cdots n)$. This algebra Π_{2n} is also generated by $2n$ p's and q's defined as follows which appear in the theory of the second quantization or in the spinor theory[5].

$$\begin{aligned} p_k &= s_k c_{k-1} c_{k-2} \cdots c_2 c_1 \\ q_k &= c_1 c_2 \cdots c_{k-1} (i c_k s_k). \end{aligned} \qquad (k=1, 2, \cdots n)$$

These operators satisfy the simple commutation relations.

$$[p_i, q_k]_+ = 0, \quad [q_i, q_k]_+ = 2\delta_{ik}, \quad [p_i, p_k]_+ = 2\delta_{ik},$$

where $[A, B]_+$ denotes $AB+BA$.
Or in a single relation

$$\Big[\sum_{k=1}^{n} (x_k p_k + y_k q_k)\Big]^2 = \sum_{k=1}^{n} (x_k^2 + y_k^2)$$

where x_k and y_k are any complex numbers. This relation suggests that if we

carry the linear orthogonal transformation in (p, q) system, the above relation remains unchanged. So that this transformation gives an automorphism of the algebra Π_{2n}. If we define $C=C_1 C_2 C_3 \cdots C_n$ which reverses all spins at once, then C anticommutes with all p's and q's and commutes with the products of even number of these. And as

$$c_k = i p_k q_k, \quad s_k s_{k+1} = -i p_{k+1} q_k$$

C also commutes with V. If we extend the suffices of p and q over n

$$p_{i+n} = -C p_i, \quad q_{i+n} = C \cdot q_i, \quad p_{i+2n} = p_i, \quad q_{i+2n} = q_i.$$

Hence p_i and q_i have period $2n$ in their suffices. Making use of these, we define the following operators which are invariant or alternating under the transformation $p_k \rightarrow p_{k+1}$, $q_k \rightarrow q_{k+1}$;

$$A_m = \frac{1}{2} \sum_{k=1}^{2n} p_{k+m} q_k = \sum_{k=1}^{n} p_{k+m} q_k,$$

$$A_m' = \frac{1}{2} \sum_{k=1}^{2n} p_{k+m} q_k (-)^k = \sum_{k=1}^{n} p_{k+m} q_k (-)^k.$$

Then V is written as

$$V = (2 \operatorname{sh} 2H)^n \exp\{-iH(A_1 + A_1')/2\} \exp\{iH^* A_0\}$$
$$\times \exp\{-iH(A_1 - A_1')/2\} \exp\{iH^* A_0\}.$$

As A_m nnd A_m' commute with C, V also commute with C. Then V lies in the subalgebra G of Π_{2n} which commute with C which accordingly divided into two ideals $G(1+C)/2$ and $G(1-C)/2$. Here we shall define $2n$ (ξ, η) as the linear orthogonal transformation of (p, q)

$$\xi_0 = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} p_k, \qquad \xi_r = \sqrt{\frac{2}{n}} \sum_{k=1}^{n} \cos \frac{r\pi}{n} k \cdot p_k,$$

$$\eta_0 = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} q_k, \qquad \xi_r^+ = \sqrt{\frac{2}{n}} \sum_{k=1}^{n} \sin \frac{r\pi}{n} k \cdot p_k,$$

$$\xi_n = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} (-)^k p_k, \qquad \eta_r = \sqrt{\frac{2}{n}} \sum_{k=1}^{n} \cos \frac{r\pi}{n} k \cdot q_k,$$

$$\eta_n = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} (-)^k q_k, \qquad \eta_r^+ = \sqrt{\frac{2}{n}} \sum_{k=1}^{n} \sin \frac{r\pi}{n} k \cdot q_k.$$

Here $\quad r=0, 2, 4 \ldots\ldots\ldots 2n-2 \quad$ or

$\quad r=1, 3, 5 \ldots\ldots\ldots 2n-1,$

We take exclusively even or exclusinely odd series from 0 to $2n-1$. Then as $\xi_{-r} = \xi_{2n-r} = \xi_r$, $\xi_{n+r} = \xi_{n-r}$, $\xi_{-r}^+ = -\xi_r^+$, $\xi_{n+r}^+ = -\xi_{n-r}^+$ and similarly for η and η^+, there are $2n$ linearly independent (ξ, η) in both cases. The commutation rela-

tions are summarized to

$$[\sum(x_r\xi_r+x_r^+\xi_r^++y_r\eta_r+y_r^+\eta_r^+)]^2=\sum(x_r^2+x_r^{+2}+y_r^2+y_r^{+2}).$$

where summation extend from 0 to n (even series) or from 1 to $n-1$ (odd series), and x_r, x_r^+, y_r, y_r^+ are any complex numbers. Namely, these $2n$ (ξ, η) are anticommuting set isomorphic to (p, q) system. So that the algebra generated by these (ξ, η) is isomorphic to Π_{2n} which is generated by (p, q). Or in other word, this transfoimation gives the automorphism of the algebra Π_{2n}. In these (ξ, η) system too, C has the invariant character.

Since $\sum p_kq_k=\sum(\xi_r\eta_r+\xi_r^+\eta_r^+)$, it follows

$$\Pi(p_kq_k)=\exp\frac{\pi}{2}\sum p_kq_k=\exp\frac{\pi}{2}(\sum\xi_r\eta_r+\xi_r^+\eta_r^+)=\Pi(\xi_r\eta_r)(\xi_r^+\eta_r^+).$$

So $$C=i^n\prod_{k=1}^{n}(p_kq_k)=i^n\prod_{r:\,\text{odd}}(\xi_r\eta_r)(\xi_r^+\eta_r^+)=i^n\prod_{r:\,\text{even}}(\xi_r\eta_r)(\xi_r^+\eta_r^+)_{\text{factors}}.$$

C also commute with the products of even number of (ξ, η). Here let us introduce the following operators having the propeities that they are invariant or alternant under the transformation

$$p_k\to p_{k+1},\ q_k\to q_{k+1}.\quad (k=1, 2, 3\ldots\ldots\ldots 2n)$$

$$X_r=\frac{1}{2n}\sum_{k,l=1}^{2n}\cos\frac{r\pi}{n}(k-l)\,p_kq_e=\frac{1-(-)^rC}{n}\sum_{k,l-1}^{n}\cos\frac{r\pi}{n}(k-l)\,p_kq_e$$

$$=\frac{I_r}{2}(\xi_r\eta_r+\xi_r^+\eta_r^+),\quad X_0=I_0\xi_0\eta_0,\quad X_n=I_n\xi_n\eta_n.$$

$$Y_r=\frac{I_r}{2}(\xi_r^+\eta_r-\xi_r\eta_r^+),$$

$$U_r=\frac{I_r}{2}(\xi_r\eta_{n+r}+\xi_r^+\eta_{n+r}^+)\quad U_0=I_0\cdot\xi_0\eta_n,\quad U_n=I_0\cdot\xi_n\eta_0,$$

$$V_r=\frac{I_r}{2}(\xi_r^+\eta_{n+r}-\xi_r\eta_{n+r}^+),$$

$$M_r=\frac{-1}{2n}\sum_{k,l=1}^{2n}\cos\frac{r\pi}{n}(k-l)\,p_xq_kp_xq_l(-)^l=\frac{2I_r}{n}\sum_{k,l=1}^{n}\cos\frac{r\pi}{n}(k-l)\,q_kq_l(-)^l$$

$$=\frac{I_r}{2}(\eta_r\eta_{n+r}+\eta_r^+\eta_{n+r}^+),\quad M_0=I_0\cdot\eta_0\eta_n=-M_n.$$

$$N_r=\frac{I_r}{2}(\eta_r^+\eta_{n+r}-\eta_r\eta_{n+r}^+).$$

$$M_r'=\frac{I_r}{2}(\xi_r\xi_{n+r}+\xi_r^+\xi_{n+r}^+),\quad M_0'=I_0\xi_0\xi_n=-M_n',$$

$$N_r' = \frac{I_r}{2}(\xi_r^+ \xi_{n+r} - \xi_r \xi_{n+r}^+),$$

$$j_r = I_r \xi_r^+ \xi_r, \qquad \left[I_r = \frac{1}{2}(1-(-)^r C)\right].$$

$$k_r = I_r \eta_r^+ \eta_r.$$

As seen by inspection, these operators having the suffices $(r, -r, n+r, n-r)$ contain only $(\xi_r, \xi_r^+, \eta_r, \eta_r^+, \xi_{n+r}, \xi_{n+r}^+, \eta_{n+r}, \eta_{n+r}^+)$ beside C.

These eight (ξ, η) generate the simple algebra Π_8 of order 2^8.

When $r=0$, $r+n=n$, they contain $(\xi_0, \eta_0, \xi_n, \eta_n)$ and when $r=n/2$, $r+n=3n/2$, they contain $(\xi_{n/2}, \eta_{n/2}, \xi_{n/2}^+, \eta_{n/2}^+)$, each set generating a simple algebra Π_4 of order 16.

In our algebra, (ξ, η) always appear in even products, so that we confine ourselves to the subalgebra $\boldsymbol{F}_r$ of Π_8 which commutes with C. This semi-simple algebra is of order 2^7 with the exceptional cases of $r=0$ and $r=n/2$.

Since $\boldsymbol{F}_1, \boldsymbol{F}_3, \boldsymbol{F}_5 \cdots\cdots \boldsymbol{F}_{n/2-1}$ have no common (ξ, η) and elements of one F are all commutative with o her F. So that we can consider the algebra of direct products of these. (Hereafter we assume $n/2$ is even to avoid unnecessary complexity)

$$\boldsymbol{D} = \boldsymbol{F}_1 \times \boldsymbol{F}_3 \times \cdots \times \boldsymbol{F}_{n/2-1}.$$

Similarly we consider

$$D' = \boldsymbol{F}_0 \times \boldsymbol{F}_2 \times \cdots \times \boldsymbol{F}_{n/2}.$$

$\boldsymbol{D}$ also commute with C, so that $\boldsymbol{D}$ can be divided into two ideals

$$\boldsymbol{D}(1+C)/2, \quad \boldsymbol{D}(1-C)/2.$$

The algebra $\boldsymbol{H}_r$ generated by $(X_r, U_{n+r}, M_r \ldots\ldots)$ are subalgebra of $\boldsymbol{D}(1-(-)^r C)/2$ and isomorphic to a subalgebra $\boldsymbol{H}_r$ of $\boldsymbol{F}_r$. We shall investigate the properties of $\boldsymbol{H}_r$ in next chapter. In chapter 3, we shall see, how the eigenvalue operator V will be factorised into the operators each belonging to $\boldsymbol{H}_r$.

2. The Abstract Algebra H_r

In this chapter, we shall examine the properties of the algebra $\boldsymbol{H}_r$. Omitting suffix r, and putting $r+n=s$, $n-r=-s$, for brevity, the generating elements run as follows

$$X = \frac{I}{2}(\xi\eta + \xi^+\eta^+), \qquad Y = \frac{I}{2}(\xi^+\eta - \xi\eta^+),$$

$$U = \frac{I}{2}(\xi\eta_s + \xi^+\eta_s^+), \qquad V = \frac{I}{2}(\xi^+\eta_s - \xi\eta_s^+),$$

$$M=\frac{I}{2}(\eta\eta_s+\eta^+\eta_s^+), \qquad N=\frac{I}{2}(\eta^+\eta_s-\eta\eta_s^+),$$

$$M'=\frac{I}{2}(\xi\xi_s+\xi^+\xi_s^+), \qquad N'=\frac{I}{2}(\xi^+\xi_s-\xi\xi_s^+),$$

$$X_s=\frac{I}{2}(\xi_s\eta_s+\xi_s^+\eta_s^+), \qquad Y_s=\frac{I}{2}(\xi_s^+\eta_s-\xi_s\eta_s^+),$$

$$U_s=\frac{I}{2}(\xi_s\eta+\xi_s^+\eta^+), \qquad V_s=\frac{I}{2}(\xi_s^+\eta-\xi_s\eta^+),$$

$$j=I\xi^+\xi, \qquad k=I\cdot\eta^+\eta,$$

$$j_s=I\xi_s^+\xi_s, \qquad k_s=I\cdot\eta_s^+\eta.$$

From these, we can construct the following 70 linearly independent elements

$$X,\ Y,\ k_sX,\ k_sY,\ j_sX,\ j_sY,\ j_sk_sX,\ j_sk_sY,$$

$$X_s,\ Y_s,\ kX_s,\ kY_s,\ jX_s,\ jY_s,\ jkX_s,\ jkY_s,$$

$$U,\ V,\ kU,\ kV,\ j_sU,\ j_sV,\ j_skU,\ j_skV,$$

$$U_s,\ V_s,\ k_sU_s,\ k_sV_s,\ jU_s,\ jV_s,\ jk_sU,\ jk_sV_s,$$

$$M,\ N,\ jM,\ jN,\ j_sM,\ j_sN,\ jj_sM,\ jj_sN,$$

$$M',\ N',\ k_sM',\ k_sN',\ kM',\ kN',\ K_skM',\ k_skN',$$

$$j,\ k,\ j_s,\ k_s,\ jk,\ j_sk_s,\ j_sk,\ j_sj,\ k_sk,\ jk_s,$$

$$jk_sk,\ j_sk_sk,\ j_sjk,\ j_sjk_s,\ jj_skk_s,\ I,$$

$$XX_s,\ YY_s,\ XY_s,\ X_sY,\ U_sU+V_sV,\ U_sV-UV_r$$

1) The unity of $\boldsymbol{H}_r$ is $I_r=(1-(-)^rC)/2$.

2) Commutable subalgebra

$j,\ k,\ j_s,\ k_s$ are commutable each other and

$$j^2=k^2=j_s^2=k_s^2=-I_r.$$

3) Quaternion subalgebra

a) $E=(I+jk)/2,\quad X,\quad Y,\quad XY=(j-k)/2.$

b) $E'=(I+jk_s)/2,\quad U,\quad V,\quad UV=(j-k_s)/2.$

c) $E''=(I+kk_s)/2,\quad M,\quad N,\quad MN=(k-k_s)/2.$

d) $E'''=(I+jj_s)/2,\quad M',\quad N',\quad M'N'=(j-j_s)/2.$

e) $E_s=(I+j_sk_s)/2,\quad X_s,\quad Y_s,\quad X_sY_s=(j_s-{}_sk)/2.$

f) $E_s'=(I+j_sk)/2,\quad U_s,\quad V_s,\quad U_sV_s=(j_s-k)/2.$

These are six quaternions, namely they satisfy the following relations

a) $X^2=Y^2=(XY)^2=-E,\ XY=-YX,\ XE=EX=X,\ YE=EY=Y,\ E^2=E.$

And similar relations for others.

And there holds following table of multiplication.

fore \ rear	X	Y	U	V	M'	N'	M	N	X_s	Y_s	U_s	V_s
j	Y	$-X$	V	$-U$	N'	$-M'$						
k	$-Y$	X					N	$-M$			$-V_s$	U_s
j_s					$-N'$	M'			$+Y_s$	$-X_s$	V_s	$-U_s$
k_s			$-V$	U			$-N$	M	$-Y_s$	X_s		

The blanc spaces give the commutable products such as kV, jX_s etc. which are linearly independent bases. Other than these are anticommutative, e.g. $jX=-Xj=Y$,, $k_sU=-Uk_s=-V$ etc.

4) i) $XM'=YN'=(U_s-jV_s)/2,\qquad M'X=N'Y=-(U_s+jV_s)/2,$

$YM'=-XN'=(V_s+jU_s)/2,\qquad M'Y=-N'X=-(V_s-jU_s)/2.$

ii) $M'U=N'V=-(X_s+jY_s)/2,\qquad UM'=VN'=(X_s-jY_s)/2,$

$N'U=-M'V=(Y_s-jX_s)/2,\qquad UN'=-(Y_s+jX_s)/2=-VM'.$

iii) $XU_s=YV_s=-(M'+kN')/2,\qquad UX_s=VY_s=(M'-kN')/2,$

$YU_s=-XV_s=-(N'-kM')/2,\qquad UY_s=-VX_s=(N'+kM')/2.$

iv) $X_sU=Y_sV=(M'-k_sN')/2,\qquad UX_s=VY_s=(-)(M'+k_sN')/2,$

$Y_sU=-X_sV=-(N'+k_sM')/2,\qquad UY_s=-VX_s=(N'-k_sM')/2.$

v) $X_sM'=-Y_sN'=-(U-j_sV)/2,\qquad M'X_s=-N'Y_s=(U+j_sV)/2,$

$Y_sM'=X_sN'=-(V+j_sV)/2,\qquad M'Y_s=N'X_s=(V-j_sU)/2.$

vi) $M'U=-N'V=(X+jY)/2,\qquad UM'=-VN'=(X-jY)/2,$

$N'U=M'V=(Y-jX)/2,\qquad UN'=VM'=-(Y+jX)/2.$

vii) $X_sU_s=Y_sV_s=(M-j_sN)/2,\qquad U_sX_s=V_sY_s=(-)(M+j_sN)/2,$

$Y_sU_s=-X_sV_s=(N+j_sM)/2,\qquad U_sY_s=-V_sX_s=-(N-j_sM)/2.$

viii) $XU=YV=(-)(M+jN)/2,\qquad UX=VY=+(M-jN)/2,$

$YU=-XV=(N-jM)/2,\qquad UY=-VX=-(N+jM)/2.$

ix) $XM=-YN=(U+kV)/2,\qquad MX=-NY=-(U-kV)/2,$

$YM=XN=(V-kU)/2,\qquad MY=NX=-(V+kU)/2.$

x) $X_sM=Y_sN=(-)(U_s+k_sV)/2,\qquad MX_s=NY_s=(U_s-k_sV_s)/2,$

$$Y_s M = -X_s N = -(V_s - k_s U_s)/2, \quad MY_s = (-)NX = +(V_s + k_s U_s)/2.$$

xi) $MU = NV = (X - k_s Y)/2, \quad UM = VN = -(X + k_s Y)/2,$

$UN = -VM = (Y - k_s X)/2, \quad NU = -MV = -(Y + k_s X)/2.$

xii) $U_s M = -V_s N = (X_s + kY_s)/2, \quad MU_s = -NV_s = -(X_s - kY_s)/2,$

$U_s N = V_s M = (Y_s - kX_s)/2, \quad NU_s = MV_s = -(Y_s + kX_s)/2.$

5) (X, Y) commute with (X_s, Y_s) and (U, V) commute with (U_s, V_s) and (M, N) commute with (M', N').

6) $UU_s - VV_s = -(XX_s - YY_s), \quad UV_s + VU_s = -(XY_s + X_s Y),$

$XX_s + YY_s = -(MM' + N'N), \quad XX_s - X_s Y = -(MN' - M'N),$

$MM' - NN' = U_s U + V_s V, \quad MN' + NM' = U_s V - UV_s.$

These complete the multiplication table of algebra $\boldsymbol{H}_r$.

In the special case of $r=0$, the algebra $\boldsymbol{H}_0$ made up of X, X_n, U, U_n, M', M, I, and $XX_n = X_n X = -UU_n = -U_n U = -MM' = -M'M$.

The multiplication table becomes

fore \ rear	X	X_n	U	U_n	M	M'
X	$-I$	XX_n	$-M$	$-M'$	U	U_n
X_n	XX_n	$-I$	M'	M	$-U_n$	$-U$
U	M	$-M'$	$-I$	$-X_n X$	$-X$	X_n
U_n	M'	$-M$	$-XX_n$	$-I$	X_n	$-X$
M	$-U$	U_n	X	$-X_n$	$-I$	$-XX_n$
M'	$-U_n$	U	$-X_n$	X	$-XX_n$	$-I$

This algebra is of order 8, and the center is composed of (I, XX_n). Then the algebra is divided into the direct sum of two simple ideals of order 4, i.e. two quaternions

$$(I - XX_n)/2, \quad (X + X_n)/2, \quad (U - U_n)/2, \quad (M - M')/2 \quad \text{and}$$

$$(I + XX_n)/2, \quad (X - X_n)/2, \quad (U + U_n)/2, \quad (M + M')/2.$$

Nextly the algebra $\boldsymbol{H}_{n/2}$ is composed of 8 elements, as the direct sum of two quaternions (X, Y, XY, E) and (U, V, UV, E').

3. Eigenvalue Problem

In the preceding chapter, we have completed the construction of Algebra $\boldsymbol{H}_r$. Here we shall resolve the eigenoperator V into the product of $\bar{V}_r$ each belonging to algebra $\boldsymbol{H}_r$. Since

$$X_r=\frac{1}{2n}\sum_{k,l=1}^{2n}\cos\frac{r\pi}{n}(k-l)\,p_k q_l=\frac{1}{2n}\sum_{m=1}^{2n}\cos\frac{r\pi m}{n}\sum_{k=1}^{2n}p_{k+m}q_k$$

$$=\frac{1}{n}\sum_{m=1}^{2n}A_m\cos\frac{r\pi}{n}m,$$

$$Y_r=\frac{1}{n}\sum_{m=1}^{2n}A_m\sin\frac{r\pi}{n}m,$$

we have

$$A_m=\sum_{r=0}^{2n-1}\left[X_r\cos\frac{r\pi}{n}m+Y_r\sin\frac{r\pi}{n}m\right].$$

Similarly

$$A_m'=\sum_{r=0}^{2n-1}\left[U_r\cos\frac{r\pi}{n}m+V_r\sin\frac{r\pi}{n}m\right].$$

Especially

$$A_0=X_0+X_1+X_2+\cdots\cdots+X_{2n-1}=2(X_1+X_2+\cdots\cdots+X_{n-1})+X_0+X_n,$$

$$A_1=2\,(X_1^*+X_2^*+X_3^*+\cdots\cdots+X_{n-1}^*)+X_0-X_n.$$

Similarly

$$A_0'=2\,(U_1+U_2+U_3+\cdots\cdots+U_{n-1})+U_0+U_n,$$

$$A_1'=2\,(U_1^*+U_2^*+U_3^*+\cdots\cdots+U_{n-1}^*)+U_0-U_n,$$

where

$$X_r^*=X_r\cos\frac{r\pi}{u}+Y_r\sin\frac{r\pi}{n},$$

$$Y_r^*=X_r\sin\frac{r\pi}{n}-Y_r\cos\frac{r\pi}{n}.$$

Similar relation holds for U_r^*, V_r^*.

Then V becomes

$$V=(2\,\mathrm{sh}\,2H)^n\exp\{-iH/2\sum_{r=0}^{2n-1}(X_r^*+U_r^*)\}\exp(iH^*\sum_{r=0}^{2n-1}X_r)$$

$$X\exp\{-iH/2\sum_{r=0}^{2n-1}(X_r^*-U_r^*)\}\exp(iH^*\sum_{r=0}^{2n-1}X_r)$$

which is factorised into the product of $\bar{V}_r$:

$$V=(2\,\mathrm{sh}\,2H)^n\,\bar{V}_0\bar{V}_1\bar{V}_2\cdots\cdots\bar{V}_{\frac{n}{2}-1}\bar{V}_{n/2}$$

where

$$\bar{V}_r=\exp\{-iH(X_r^*+X_s^*)-iH\cdot(U_r^*+U_s^*)\}\exp\{2H^*i(X_r+X_s)\}$$

$$\times \exp\{-iH(X_r^*+X_s^*)+iH\cdot(U_r^*+U_s^*)\}\exp\{2H^*i(X_r+X_s)\},$$
$$(r=1,2,3\ldots\ldots n/2-1)$$

$$\bar{V}_0=\exp\{-iH/2(X_0-X_n)-iH/2(U_0-U_n)\}\exp\{iH^*(X_0+X_n)\}$$
$$\times \exp\{-iH/2(X_0-X_n)+iH/2(U_0-U_n)\}\exp\{iH^*(X_0+X_n)\},$$

$$\bar{V}_{n/2}=\exp\{-iH(Y_{n/2}+V_{n/2})\}\exp\{2H^*iX_{n/2}\}\exp\{-iH(Y_{n/2}-V_{n/2})\}$$
$$\times \exp((2H^*iX_{n/2})\cdot$$

And

$$\frac{1+C}{2}V=(2\,\mathrm{sh}\,2H)^n\frac{1+C}{2}\cdot\bar{V}_1\bar{V}_3\bar{V}_5\cdots\cdots\bar{V}_{\frac{n}{2}-1},$$

$$\frac{1-C}{2}V=(2\,\mathrm{sh}\,2H)^n\frac{1-C}{2}\cdot\bar{V}_0\bar{V}_2\bar{V}_4\cdots\cdots\bar{V}_{n/2}.$$

Hence $(1+C)/2\cdot V$ lies in the algebra $\boldsymbol{D}(1+C)/2$ where

$$\boldsymbol{D}=\boldsymbol{F}_1\times\boldsymbol{F}_3\times\boldsymbol{F}_5\times\cdots\cdots\times\boldsymbol{F}_{\frac{n}{2}-1}.$$

And $(1-C)/2\cdot V$ lies in the algebra $\boldsymbol{D}'(1-C)/2$ where

$$\boldsymbol{D}'=\boldsymbol{F}_0\times\boldsymbol{F}_2\times\boldsymbol{F}_4\times\cdots\cdots\times\boldsymbol{F}_{n/2}.$$

And $\bar{V}_r$ lies in $\boldsymbol{H}_r$ which is isomorphic to the subalgebra $\bar{\boldsymbol{H}}_r$ of $\boldsymbol{F}_r$.

If we can find the eigenvalues of $\bar{V}_r$ in $\boldsymbol{F}_r$, by making product of these in suitable selection, we can get the eigenvalues of V in the subspace $C=1$ or $C=-1$, according as the odd series or even series are adopted.

In the next section we shall find the eigenvalues of $\bar{V}_r$. (*to be continued*)

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On the Integrability Condition in the "Super-Many-Time Theory."

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§ 1. Introduction.

As the generalization of the ordinary one-time theory, the super-many-time theory has been constructed, which enables us to calculate many problems with less ambiguities than the ordinary one, because of its covariant feature, but the results coincide with the ordinary ones.

Here, in this paper, we get a new result that the one-time theory can never deduce.

The most characteristic difference between two theories is the existence of the integrability condition, which occurs from the increasement of the number of wave equations. In fact, the number of wave equations in the super-many-time theory is continuous infinity, so that the condition for the existence of the solution of the wave equations is very severe. Concretely speaking, the condition that the wave equations have the uniquely determined solution, i. e. the integrability condition restricts the properties of the interaction Hamiltonian densities between fields.

First we discuss the possible structure of the Hamiltonian density by the leading principle, "the principle of action through a medium". Next using the results, we prove that the energy-momentum conservation law requires the integrability condition in its proof. This signifies that the integrability condition is necessary not only for mathematical reason, but also for physical. Lastly as a concrete application of this condition, we deduce rule for the choice of various types of decays, which is the one that the one-time theory can never conclude.

§ 2. General type of Hamiltonian density.

Our first problem is the determination of the interaction Hamiltonian density between fields, using the above mentioned principle.

Now we start from the following wave equation:

$$\left\{H(P)+\frac{\hbar}{i}\frac{\delta}{\delta C_P}\right\}\Psi[C]=0, \tag{2.1}$$

where $H(P)$ is a function of a point P, and in general is a functional of the

space-like hypersurface C.

Consider an arbitrary field quantity $S(P)$ which is a function of a point P, and write the expectation value $S(P)$ in the state $\Psi[C]$ as $<S(P)>_C$ for simplicity, i. e.:

$$<S(P)>_C=(\tilde{\Psi}[C],\ S(P)\Psi[C]). \tag{2.2}$$

Here we interpret the principle of action through a medium as follows:
(I) Let a pair of points P and Q on the space-like hypersurface C lie a finite distance apart from each other. Then the expectation value of $S(Q)$ is not affected by infinitesimal deformation of the hypersurface C at the point P, i. e.:

$$\frac{\delta}{\delta C_P}<S(Q)>_C=0. \tag{2.3}$$

Comparing (2.3) with (2.1), one obtains

$$\begin{aligned}\frac{\delta}{\delta C_Q}<S(P)>_C&=\frac{\delta}{\delta C_Q}(\tilde{\Psi}[C],\ S(P)\Psi[C])\\&=(i/\hbar)\{(\tilde{\Psi}[C],\ H(Q)S(P)\Psi[C])-(\tilde{\Psi}[C],\ S(P)H(Q)\Psi[C])\}\\&=(i/\hbar)<[H(Q),\ S(P)]>_C=0.\end{aligned}$$

While the wave function $\Psi[C]$ is arbitrarily chosen, so that not only the expectation value but also the poisson bracket itself must vanish,

$$[H(Q),\ S(P)]=0. \tag{2.4}$$

Except the hypothesis (I), taking the foregoing principle for time direction into account, it will be natural to assume:
(II) $H[P,\ C]$ depends only on the field quantities on the surface C.

Now let us examine what kind of conclusion can be deduced, combining the hypotheses I and II.

First $H[P,\ C]$ cannot contain any field quantity of a point which lies at a finite distance apart from the point P and outside the lightcone of P, as $S(P)$ in (2.4) is an arbitrary field quantity, and so it contains only the field quantities of such points that are on the surface C and are adjacent to the point P.

Thus our first conclusion is obtained.

(A) $H[P,\ C]$ is determined by the field quantities of points which are adjacent to P and on the surface C.

Next let us try to formulate this conclusion, and we assume a mathematical property of $H[P,\ C]$.
(III) $H[P,\ C]$ is an analytic functional of the surface C, i. e.:

$$H[P,\ C]=h_0(P)+\int^{C} h(P,\ X)d\Omega+\iint^{C\,C'} h(P,\ X,\ X')d\Omega\, d\Omega'+\cdots, \tag{2.5}$$

where $d\Omega = dx^1dx^2dx^3dx^4$, and which corresponds to the hypothesis that $H[P, C]$ has the functional derivative of any order. But by the hypothesis (II), $H[P, C]$ must be expressed by surface integrals, i. e.:

$$H[P, C] = h_0(P) + \int_C h_\mu(P, X) dF_\mu + \iint_{CC} h_{\mu\mu'}'(P, X, X') dF_\mu dF'_{\mu'} + \cdots \tag{2.6}$$

The functions h's must vanish when any one of the points $X, X', \cdots$ lies at a finite distance apart from the point P because of the conclusion (A), so that the integrands vanish unless the points $X, X', \cdots$ belong to the infinitesimal vicinity of the point P. For this reason, the integrals in (2.6) vanish unless the integrals contain δ-functions, and the method to let δ-functions be contained in (2.6) comes into question. In this case, remember the following formula:

$$\left(\frac{\partial D(X_1 - X_2)}{\partial X_1^\mu}\right)_{X_1^4 = X_2^4} = \left(0,\ 0,\ 0,\ \frac{1}{ic}\delta(X_1 - X_2)\right) \tag{2.7}$$

putting the four vector as $\Delta_\mu(X_1 - X_2)$, where X_1, X_2 lie outside of each other's lightcones, then the problem is solved as follovs:

$$\begin{aligned} &h_\mu(P, X) = h_{\mu\nu}(P)\Delta_\nu(P - X), \\ &h_{\mu\mu'}(P, X, X') = h_{\mu\mu'\nu\nu'}(P)\Delta_\nu(P - X)\Delta_{\nu'}(P - X'), \text{ etc.} \end{aligned} \tag{2.8}$$

So letting (2.8) into (2.6), we get

$$\begin{aligned} H[P, C] = h_0(P) &+ \int_C h_{\mu\nu}(P)\Delta_\nu(P - X)\, dF_\mu \\ &+ \iint_{CC} h_{\mu\mu'\nu\nu'}(P)\Delta_\nu(P - X)\Delta_{\nu'}(P - X')\, dF_\mu dF_{\mu'}' \\ &+ \cdots = h_0(P) + h_{\mu\nu}(P) N_\mu N_\nu + h_{\mu\mu'\nu\nu'}(P) N_\mu N_{\mu'} N_\nu N_{\nu'} + \cdots \end{aligned} \tag{2.9}$$

where N_μ is the unit normal vector of C at P.

Thus we get the second conclusion:

(B) The general form of the interaction Hamiltonian density is given by the equation (2.9).

We confine ourselves to a more special case, thinking physically of this problem. For this purpose we prove the following lemma.

Lemma. If a functional $F[P, C]$ of the form (2.9) vanishes for any flat space-like surface C_0, then it vanishes for any space-like surface C. Proof. Let the tangential hyperplane of C at P be C_0, then C_0 is flat of course, so from the special form of (2.9) we can obtain at once that $F[P, C]$ is equal to $F[P, C_0]$. But as $F[P, C_0] = 0$ from the antecedents, we get

$$F[P, C] = 0.$$

Well, we know in the one-time-theory that the Hamiltonian density must be of the form

$$(\text{scalar}) + (\text{tensor})_{44}.$$

We scalarize it as

$$(\text{scalar}) + (\text{tensor})_{\mu\nu} N_\mu N_\nu.$$

Let the latter be $H_0[P, C]$, then $H_0[P, C]$ coincides with that of one-time-theory in the case when C is flat. Now put

$$F[P, C]=H[P, C]-H_0[P, C],$$

then $F[P, C]$ satisfies the conditions mentioned in the above lemma, by which we get immediately

$$F[P, C]=0, \quad \text{i. e.} \quad H[P, C]=H_0[P, C].$$

Thus we get the third conclusion:

(C) The general form of Hamiltonian density is

$$H[P, C]=h_0(P)+h_{\mu\nu}(P)N_\mu N_\nu. \tag{2.10}$$

This conclusion signifies that we can obtain $H[P, C]$ only by the scalarization of the one-time theoretical one. Here, what we must notice is the fact that the local Hamiltonian density in the one-time theory cannot be determined uniquely in general. In order to avoid this difficulty, we shall be obliged to test by the conclusion (D) which is given next.

Usually $H[P, C]$ is obtained by solving the following equation named integrability condition of the generalized Schrödinger equation (2.1):

$$\left[H[P, C]+\frac{\hbar}{i}\frac{\delta}{\delta C_P},\ H[Q, C]+\frac{\hbar}{i}\frac{\delta}{\delta C_Q}\right]=0,$$

$$\frac{\hbar}{i}\left(\frac{\delta H[Q, C]}{\delta C_P}-\frac{\delta H[P, C]}{\delta C_Q}\right)+[H[P, C],\ H[Q, C]]=0. \tag{2.11}$$

We interprete the solution by this method as follows:

(D) The equation (2.11) gives the criterion for the correctness of the choice of the interaction Hamiltonian density $H[P, C]$.

Therefore we can assume that the mathematical condition (2.11) is satisfied hereafter.

§ 3. Conservation law of energy and momentum.

In this section we prove the conservation law of energy and momentum in the super-many-time theory, making use of the results obtained in the previous section. We start from mathematical preliminaries.

Formula 1.

$$\frac{\delta}{\delta C_Q}(N_\mu N_\nu)_P = \frac{\partial^2 D(Q-P)}{\partial x_\mu' \partial x_\nu'}. \tag{9.1}$$

Proof.

$$\int^C \frac{\partial^2 D(Q-P)}{\partial x_\mu' \partial x_\nu'} d\Omega' = \int_C N_\nu' \frac{\partial D(Q-P)}{\partial x_\mu'} dF',$$

where dashes for argument Q, N the normal of C to future,

$$= \int_C N_\nu' N_\mu \delta(Q-P) dF' = N_\mu N_\nu.$$

Formula 2. Let the canonical energy-momentum tensor of the free fields in the interaction representation be $T_{\mu\nu}$, and put

$$\mathfrak{P}_\nu^0 = \int_C T_{\mu\nu}\, dF_\mu, \tag{3.2}$$

then the following formula is obtained:

$$\left[\mathfrak{P}_\nu^0, S\right] = \frac{\hbar}{i} \frac{\partial S}{\partial x_\nu}, \tag{3.3}$$

where S is a point field quantity.

Proof. The interaction representation is equivalent to the Heisenberg representation of the free fields, so that it remains correct in our case.

Next we will define the total differential by

$$\frac{\delta}{\delta C_P}(\tilde{\Psi}[C],\ S\Psi[C]) = (\tilde{\Psi}[C], \frac{\Delta S}{\Delta C_P}\Psi[C]), \tag{3.4}$$

or explicitly

$$\frac{\Delta S}{\Delta C_P} = \frac{\delta S}{\delta C_P} + \frac{i}{\hbar}[H(P),\ S] = \left[\frac{\delta}{\delta C_P} + \frac{i}{\hbar} H(P),\ S\right] \tag{3.5}$$

The total is the functional differential when we employ for the physical quantities, Heisenberg representation in place of the interaction representation, so it is very natural to define the conservative quantity S by

$$\frac{\Delta S}{\Delta C_P} = 0. \tag{3.6}$$

From this definition, it is easily proved that the set of all the conservative quantities form a Lie ring. If a quantity is conservative, then so is in the one-time-theory. The converse, however, is not always true, and this is the reason why it is necessary to prove that energy and momentum conserve, which is firmly

structed in the one-time theory.

Now the total energy and momentum of the fields in the interaction representation are defined by

$$\mathfrak{P}_\nu=\int_C T_{\mu\nu}dF+\int_C H[X, C]dF_\nu=\mathfrak{P}_\nu^0+\int_C H[X, C]dF_\nu, \tag{3.7}$$

where

$$H[X, C]=h_0(X)+h_{\mu\nu}(X)N_\mu N_\nu. \tag{2.10}$$

So the task we must finish is the proof of (3.6) for $\mathfrak{P}_\nu$.

$$\frac{\Delta\mathfrak{P}_\nu}{\Delta C_P}=\frac{\partial\mathfrak{P}_\nu}{\delta C_P}+\frac{i}{\hbar}[H(P), \mathfrak{P}_\nu]$$

$$=\frac{\partial T_{\mu\nu}}{\partial x_\mu}+\frac{\delta}{\delta C_P}\int_C H(X)dF_\nu+\frac{i}{\hbar}\int_C [H(P), H(X)]dF_\nu+\frac{i}{\hbar}\int_C [H(P), T_{\mu\nu}(X)]dF_\mu$$

$$=\frac{\delta}{\delta C_P}\int_C (h_0(X)+h_{\alpha\beta}(X)\bar{N}_\alpha\bar{N}_\beta)dF_\nu+\frac{i}{\hbar}\int_C [H(P), H(X)]dF_\nu+\frac{i}{\hbar}\int_C [H(P), T_{\mu\nu}(X)]dF_\mu$$

$$=\left\{\frac{\partial h_0}{\partial x^\nu}+\frac{\partial h_{\alpha\beta}}{\partial x_\nu}N_\alpha N_\beta+\int_C h_{\alpha\beta}(X)\frac{\delta}{\delta C_P}(\bar{N}_\alpha\bar{N}_\beta)dF+h_{\alpha\beta}\int_C \frac{\partial^2 D(P-X)}{\partial x_\alpha\partial x_\beta}dF_\nu\right\}$$

$$+\frac{i}{\hbar}\left\{\int_C [H(P), H(X)]dF_\nu+\int_C ([h_0(P), T_{\mu\nu}(X)]+[h_{\alpha\beta}(P), T_{\mu\nu}(X)\bar{N}_\alpha\bar{N}_\beta])dF_\mu\right\}$$

$$=\left\{\frac{\partial h_0}{\partial x_\nu}+\frac{\partial h_{\alpha\beta}}{\partial x_\nu}N_\alpha N_\beta+\int_C \frac{\partial H(X)}{\delta C_P}dF_\nu+h_{\alpha\beta}\int_C \frac{\partial^2 D(P-X)}{\partial x_\alpha\partial x_\beta}dF_\nu\right\}$$

$$+\left\{\frac{i}{\hbar}\int_C [H(P), H(X)]dF_\nu-\frac{\partial h_0}{\partial x_\nu}-\frac{\partial h_{\alpha\beta}}{\partial x_\nu}N_\alpha N_\beta\right\}$$

$$=\frac{\hbar}{i}\int_C \left[\frac{\delta}{\delta C_P}+\frac{i}{\hbar}H(P), \frac{\delta}{\delta C_X}+\frac{i}{\hbar}H(X)\right]dF_\nu, \tag{3.8}$$

where N_α, $\bar{N}_\alpha$ are normals at the points P, and X respectively, and point function with no argument signifies the function of the point P. Remembering the integrability condition, we get consequently

$$\frac{\Delta\mathfrak{P}_\nu}{\Delta C_P}=0. \tag{3.9}$$

This is just the conservation law of energy and momentum in the case of the super-many-time theory. Thus we get:

(E) The integrability condition is sufficient for energy-momentum conservation law, moreover it will be necessary at the same time.

§ 4. Restriction rule for decay types.

In the previous section we saw that the integrability condition has a physically important character. So let us derive a conclusion from this condition. If a pair of points P and Q on a space-like hypersurface C lie a finite distance apart from each other, then by the conclusion (C) obtained in § 2, we know that

$$\frac{\delta}{\delta C_P} H[Q, C] = \frac{\delta}{\delta C_Q} H[P, C] = 0. \tag{4.1}$$

So in this case the integrability condition (2.11) or (4.2) :

$$\left[H[P, C] + \frac{\hbar}{i} \frac{\delta}{\delta C_P},\ H[Q, C] + \frac{\hbar}{i} \frac{\delta}{\delta C_Q} \right] = 0 \tag{4.2}$$

reduces to the following simple form:

$$[H(P), H(Q)] = 0. \tag{4.3}$$

We confine ourselves to this special case for the situations of the points P and Q in this section, therefore the derived results are all necessary conditions, but not sufficient in general.

Now we decompose the Hamiltonian $H(P)$ into several parts, each part corresponding to an elementary process, and then we will classify them into two classes.

$$H(P) = \sum H_s(P).$$

Modification for $H(P)$ to let satisfy the integrability condition should be taken into account in the determination of the individual Hamiltonians as was discussed in (D). We introduce a notation for the elementary process. For example, the emission of a photon by an electron, i. e.:

$$e \rightarrow e + \gamma, \tag{4.4}$$

includes two electrons and one photon in the process.

This process shall be denoted by

$$(\gamma^1;\ e^2). \tag{4.5}$$

Generally any elementary process can be represented by the notation as

$$(B_1^{m_1},\ B_2^{m_2}, \cdots;\ F_1^{n_1},\ F_2^{n_2}, \cdots), \tag{4.6}$$

where B_1, $B_2, \cdots$ for Bose particles, and F_1, $F_2, \cdots$ for Fermi particles. From the spin conservation law, we know easily that

$$n_1 + n_2 + \cdots \equiv 0, \quad (\text{mod. } 2), \tag{4.7}$$

for any elementary process. We write (4.6) simply as

$$(n_1, n_2, \cdots).$$

Then two cases occur, when

(a) $n_1 \equiv n_2 \equiv \cdots \equiv n_s \equiv 0$, (mod. 2),

(b) any other case.

We call (a) and (b) as "even type", "odd type" respectively.

Now let ϕ_α and ψ_β be Bose field wave function and Fermi one respectively, then in our case we can write simply

$$[\phi_\alpha(P),\ \phi_{\alpha'}(Q)]=0,$$
$$\{\psi_\beta(P),\ \psi_{\beta'}(Q)\}=0. \tag{4.8}$$

Moreover, wave functions of different fields always commute with each other, so we see at once that

$$[H_+(P),\ S(Q)]=0, \tag{4.9}$$

where $H(P)$ is a Hamiltonian of the even type, and $S(Q)$ any point field function, Thus the condition (3.3) reduces to

$$[H_-(P),\ H_-(Q)]=0. \tag{4.10}$$

where, H_+ and H_- are even and odd part of H respectively. And we can neglect the discussions concerning the even part, so we will consider about the commutation relations between odd type Hamiltonians. Suppose that

$$\begin{aligned} H_s &\sim (n_1,\ n_2 \cdots), \\ H_{s'} &\sim (n_1',\ n_2', \cdots), \end{aligned} \tag{4.11}$$

then by the commutation relations (4.8) we get readily

$$H_s(P)H_s'(Q) = \varepsilon H_s'(Q)H(P),$$

where

$$\varepsilon=(-1)^{n_1n_1'+n_2n_2'+\cdots\cdots}. \tag{4.12}$$

So that the necessary condition for the vanishing of $[H_s(P), H_s'(Q)]$ is given by the following equation:

$$n_1n_1'+n_2n_2'+\cdots \equiv 0, \qquad \text{(mod. 2)}. \tag{4.13}$$

Two types of decays which do not satisfy (4.13), i.e.

$$n_1n_1'+n_2n_2'+\cdots \equiv 1, \qquad \text{(mod. 2)}. \tag{4.14}$$

shall be called "incompatible pair".

Suppose that $H_-(P)$ is expressed by

$$H_-(P)=\sum H_s(P), \tag{4.15}$$

then the commutation relation (4.10) requires

$$[H_s(P),\ H_s'(Q)]=0, \tag{4.16}$$

for each pair of elementary processes because of the independence of the wave functions. Thus from (4.16) and (4.13) we can conclude a restriction rule for decays.

(F) By the super-many-time theory, incompatible pair of decays can never exist.

As the application of this conclusion (F), let us test the compatibility of meson decays. Now consider

$$\pi^- \to \mu^- + \nu, \tag{i}$$

$$\mu^- \to e^- + \nu + \nu, \tag{ii}$$

then we see that

$$\sum nn' = \underbrace{1\times 1}_{\mu} + \underbrace{0\times 1}_{e} + \underbrace{1\times 2}_{\nu} \equiv 1, \quad (\text{mod. } 2)$$

therefore (i) and (ii) are incompatible.

In order to improve this incompatibility, the following modification of (ii) is considered

$$\mu^- \to \mu_0 + e^- + \nu, \tag{ii$'$}$$

where μ_0 stands for a neutral μ-meson. For the combination of (i) and (ii$'$)

$$\sum nn' = \underbrace{1\times 1}_{\mu} + \underbrace{0\times 1}_{\mu_0} + \underbrace{0\times 1}_{e} + \underbrace{1\times 1}_{\nu} \equiv 0, \quad (\text{mod. } 2).$$

In such a case, we can make another modification which is of a more general type, that the wave functions of some different kinds of Fermi particles must anticommute with each other in contrast with the ordinary one that they must always commute. This is a generalization of Pauli's exclusion principle, and the particles with anticommuting wave functions have to be regarded as the different states of the same kind of a Fermi particle just as the proton and the neutron states of a nucleon.

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The Electronic Component in Extensive Air Showers

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§ 1. Introduction

The extensive air shower is a phenomenon with the greatest energy in nature, and thus it is of much interest to see whether the current theory is still valid in such an extremely high energy region. We have seen in our previous analysis of cosmic-ray underground[1] and bursts,[2, 3] that the quantum electrodynamics and the model of mesons are valid up to about 10^{12} eV. To advance our study to much higher energy region, we revise our previous work[4] under the light of the recent development on this problem.

In this paper, we treat mainly the behaviour of the electronic component, which is the main part of an air shower. Calculations are made for the density-frequency relation, the altitude variation and the directional distribution. These characters of a shower will mainly concern with the electromagnetic process, and should be examined first, since the electrodynamics is supposed to be based on more reliable ground than others even in such a high energy region. The lateral structure of a shower, especially in the vicinity of its core, will have to do with the nuclear interactions, and will be treated in the subsequent paper.

The most works performed by various authors on this problem have assumed the electron primary hypothesis. Although they have got much success to explain the general behaviour of the electronic component, this may not always be considered as the proof of the electron primary hypothesis[5]. This hypothesis can be disproved, for example, by considering the structure of a shower core,[6] as will be discussed in detail in the subsequent paper. Furthermore, the behaviour of the penetrating component in a shower can be well understood only by assuming protons as well as heavy nuclei as primary rays. Thus we put forward the analysis based on the proton primary hypothesis and inquire about the origin of the electronic component.

First we will estimate the fraction of the energy of the primary proton, which is given to the electronic radiation (§ 2). We consider the two plausible processes for the production of the electronic rays in nuclear collision, one of which is the γ-decay of a neutral meson and the other is the charge exchange process*). Then

*) It is regrettable that most of American authors misunderstand the charge exchange process, which is only effective in the high energy region, but hardly effective in so low energy as the production of artificial mesons.

we calculate the cascade development of a shower analytically as far as possible, adopting the appropriate magnitude of the cross-section (§ 3). The size-frequency relation and altitude variation are obtained from this calculation, leaving some physical parameters undetermined (§ 4). Finally, we check the error in the above approximate treatment by the numerical calculations. Numerical results are compared with the recent experimental data (§ 5).

§ 2. The Energy Transported into Electronic Rays.

One may imagine that the primary proton with the given energy will make rather smaller shower than the primary electron with the same energy, since the primary proton can give only a fraction of its energy to the electronic component. This will be true for a shower at its maximum position, where the size of a shower is approximately determined only by the ratio of the energy given to the electronic component to the critical energy of air. Here, we must remark the following points:

1) Most of the air showers observed at the lower altitude, however, are supposed to have already passed their maxima, if they originate from primary electrons.

2) Primary protons will emit electronic radiation by the nuclear collision at the atmospheric depth of the order of the mean free path for collision.

3) Almost all the authors have referred the primary spectrum adopted by Heisenberg[8], which has turned out to give much smaller intensity than the recent experimental value.

These effects may make the primary proton hypothesis to have sufficient frequency of showers against the above supposition. To see this situation, we must estimate first how much energy the primary proton gives to the electronic component.

It is experimentally confirmed that the nucleon component decreases as exp $(-l/\Lambda)$ where the absorption mean free path Λ is $\sim 125 \mathrm{g/cm^2}$. This may allow us to assume the following exponential decrease of the energy E of a primary proton through the atmosphere, considering that the energy spectrum of the nucleon component does not seem to change drastically.

$$E=E_0 \exp(-al). \tag{1}$$

The absorption coefficient a is connected with the above meen free path Λ by

$$a=1/\Lambda\lambda \tag{2}$$

where λ is the index of the primary power spectrum, ~ 1.8. The above simplification neglects the multiplication of the nucleon component, but may be a good approximation as far as we are concerned only with the energy transported into electronic rays.

For later convenience, we use the radiation unit of air, 36g/cm^2, as the unit of atmospheric depth. Then the numerical value of a is 0.160. All the energies are measured in the critical energy of air $\varepsilon_j=86$ MeV. Furthermore, we use the following variable Y_0 instead of the energy E_0.

$$Y_0=ln\,(E_0/\varepsilon_j) \tag{3}$$

The primary energy spectrum is written as

$$F(Y_0)=A\ \exp\ (-\lambda Y_0) \tag{4}$$

As for the possible mechanisms of the production of electronic rays, we consider the γ-decay of a neutral meson and the charge exchange process.

The charge exchange process gives the probability per radiation unit for photon production as follows[7],

$$\begin{aligned}&\sigma(Y,\,y)dy=\sigma_0 kYdy,\\&\sigma_0=a_0\left(\frac{2A^{1/3}}{137\pi}\right)=6.20\times10^{-3}.\end{aligned} \tag{5}$$

Y and y correspond to the energy of the primary nucleon and of the emitted photon, respectively. k is the probability for charge exchange in a nuclear collision and is left undetermined. a_0 is the mean free path for a nuclear collision which we assume here to occur with the geometrical cross-section of an air nucleus. Now we estimate the energy transported to the electronic rays through the charge exchange process as,

$$\int_0^\infty e^{Y_0-al}\sigma_0 k(Y_0-al)\,dl\approx3.9\times10^{-2}k(Y_0-1), \tag{6}$$

For the primary energy 10^{15} eV, i.e. $Y_0\approx16$, this becomes about $58k\%$.

While, we know only little about the production rate and the life of the neutral mesons: The analysis on the μ-meson spectrum[9] shows, that the contribution from the neutral meson is the same order of magnitude as (6) putting $k=1$, if we assume the sufficient short life and the symmetrical meson theory.

§ 3. The Size of Extensive Air Showers.

We will calculate the cascade development of a shower from the charge exchange process, for we can give the explicit expression of the cross-section only for this process.[10]

From (1) we can express the decrease of the energy of a proton as,

$$Y=Y_0-al. \tag{7}$$

The probability that this proton produces a photon with energy between y and $y+dy$ is given by (5). Then the size of a shower at depth l is obtained as,

$$N(Y_0, l) = \sigma_0 k \int_0^l dl' \{Y_0 - a(l-l')\} \int_0^{Y_0 - a(l-l')} dy\, C(y, l'), \tag{8}$$

where $C(y, l')$, the cascade function, means the number of electrons in a shower with incident energy y after traversing l'.

In order to evaluate the integral (8), we make use of the following character of the cascade function. The value of $C(y, l')$, as represented in Fig. 1, has a peak along a line $y=l'$. The integral region of (8) is surrounded by two coordinate axis and two lines, $y=Y_0-a(l-l')$ and $l'=l$. We shall modify this region and change the order of integration.

For $Y_0 \ll l$, we may replace the boundary line $y=Y_0-a(l-l')$ by the line which is parallel to l'-axis and passes through the crossing point of $y=Y_0-a(l-l')$ and $y=l'$. The coordinate of this crossing point is

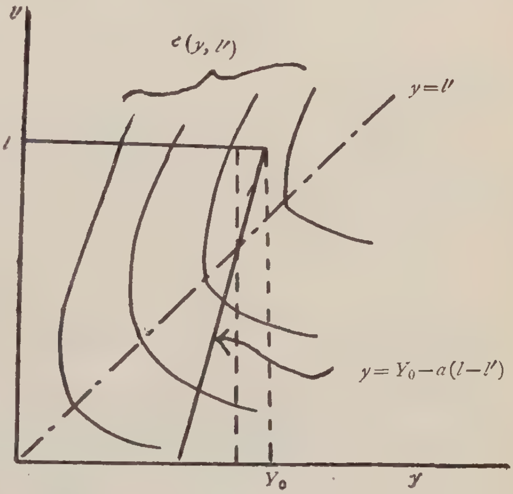

Fig. 1. The modification of the integral region; ——: exact bonudary, ----: modified boundary

$$y = l' = Y_0/(1-a) - al/(1-a) = 1.190\, Y_0 - 0.195 l. \tag{9}$$

Then we can first integrate over l'. Here we may extend the upper limit of the integration to infinity, since the upper limit l lies beyond the maximum point of the integrand.

The error introduced by this approximation can be estimated as follows. For simplicity, we refer to the cascade theory of Heisenberg[8], in which the ionization loss of an electron is neglected. The integral from 0 up to infinity is

$$\int_0^\infty dl' C(y, l') = 0.437 \exp y. \tag{10}$$

While the integral to l_{max}, which corresponds to the maximum value of $C(y, l)$, is

$$\int_0^{l_{max}} dl' C(y, l') \approx \frac{1}{2} C(y, l_{max}) l_{max} = \frac{0.13}{2} \sqrt{1.02\, y} \exp y. \tag{11}$$

Here we use the relations $C(y, l_{max}) = 0.13 \exp y/\sqrt{l_{max}}$ and $l_{max} = 1.02\, y$. For $y_0 = 16$,

$$(10)/(11) \approx 1.4. \tag{12}$$

Thus our approximation is the over estimation at most 40%.

Once our approximation is justified, we perform the integral (8), using the

more exact formulas[11], which include the effect of ionization loss,

$$\int_0^\infty dl' C(y, l') = \exp y, \tag{13}$$

and

$$\int_0^\infty dl' l' C(y, l') = (1.01\, y + 0.4) \ \exp y. \tag{14}$$

Then we get

$$N(Y_0, l) = \sigma_0 k \left\{ \frac{Y_0 - al}{1-a} - 0.61a \right\} \ \exp \left\{ \frac{Y_0}{1-a} - \frac{a}{1-a} l \right\}. \tag{15}$$

Inserting the numerical value for a, this formula is approximated by

$$N(Y_0, l) = \exp \{ 1.19\, Y_0 - 0.19 l - 3.0 \}, \tag{16}$$

while in the case of the electron primary, we get the corresponding formula

$$N(Y_0, l) = \exp \left\{ 2\sqrt{y_0 l} - l - \frac{1}{2} ln\, l - 1.4 \right\}. \tag{17}$$

Comparing both formulas, we see they are in agreement within the factor 2 in the neighbourhood of $l=20$. Thus our expectation mentioned in § 2 is proved.

§ 4. Size-Frequency Relation and Altitude Variation.

We obtain the density of electrons at the distance r from the shower axis, by multiplying the size $N(Y_0, l)$ with Moliere function $M(l, r)$,

$$\varDelta(Y_0, l, r) = N(Y_0, l) M(l, r). \tag{18}$$

Here we assume that the functional form of $M(l, r)$ does not depend on the shower age and adopt the analytical form derived by Bethe[2],

$$M(l, r) = (0.45/r\bar{r})\,(1 + 4r/\bar{r}) \ \exp \{ -4(r/\bar{r})^{2/3} \}. \tag{19}$$

The average spread $\bar{r}$ depends on l as,

$$\bar{r} = r_0 (28.6/l). \tag{20}$$

$r_0 = 74$m is the value of $\bar{r}$ at see level, $l=28.6$.

Introducing (15) into (18), we get

$$\varDelta(Y_0, l, r) = \sigma_0 k \{ (Y_0/P) - 0.61a - a'l \} \ \exp \ (-a'l + Y_0/P) M(l, r), \tag{21}$$

where we use the abbreviations

$$\left. \begin{aligned} a/(1-a) &= a' = 0.195, \\ 1-a &= P = 0.860. \end{aligned} \right\} \tag{22}$$

To solve Y_0 in (21) as a function of $\varDelta$, l and r, a slowly varying part $(Y_0/P) - 0.61\, a$ is replaced by the value at $l^* = 20$ and we get,

$$e^{Y_0}=\{e^{Pa'l}/(\sigma_0 k)^P(l^*-a'l)^P\}(\Delta/M)^P. \tag{23}$$

Then we obtain the integral density spectrum by substituting this relation into the primary spectrum (4),

$$F_0(Y)=A(\sigma_0 k)^\gamma(l^*-a'l)^\gamma e^{-a'\gamma l}(M/\Delta)^\gamma, \tag{24}$$

where

$$\gamma=P\lambda=1.51 \tag{25}$$

represents the index of the power density spectrum.

To compare with observations we must average over r and integrate over the falling direction of showers. For the former process we have only to evaluate the integral

$$\begin{aligned}\int_0^\infty 2\pi r dr M^\gamma &= 1.38/\bar{r}^{2(\gamma-1)}\\ &=1.38(l/28.6)^{2(\gamma-1)}/r_0^{2(\gamma-1)}.\end{aligned} \tag{26}$$

In the next integration, we neglect the variation with l in (24) except for exp $(-a'\gamma l)$, and we approximate the integral over zenith angle as

$$\begin{aligned}&2\pi\int_0^\infty \frac{ldl'}{l'^2}\ \exp\ (-a'\gamma l')\times(\text{slowly varying part})\\ &\approx\frac{2\pi}{a'\gamma l}\ \exp\ (-a'\gamma l)\times(\text{slowly varying part}).\end{aligned} \tag{27}$$

Finally we get the frequency of extensive air showers at depth l with density greater than Δ as

$$\begin{aligned}H(\Delta,l)&=2\pi A(\sigma_0 k)^\gamma(l^*-al)^\gamma l^{2\gamma-3}e^{-a'\gamma l}/\Delta^\gamma\\ &=1.35\times10^{-4}Ak^{1.51}(20.0-0.19l)^{1.51}l^{-0.024}e^{-0.288l}/\Delta^{1.51}\ hr^{-1}.\end{aligned} \tag{28}$$

Formula (28) tells us the following behaviour of extensive air showers. 1) This density spectrum $\Delta^{-\gamma}$ is in nearly good agreement with the experimental one. It means that the primary power-law spectrum (4) is fairly good. 2) The index γ of the power spectrum (26) is the constant given by (25), whereas the experimental value of this index seems to vary slightly with both density and altitude. The constancy of the γ results mainly from the approximation (22) to (23). 3) The altitude dependence is mainly governed by the factor exp $(-a'\gamma l)$ $=$ exp $(-a\lambda l)$ $=$ exp $(-l/\Lambda)$ which is the same as the absorption of primary protons. This means that the shower is in equilibrium with primary protons. Such situation will be realized only at lower altitudes.

The absolute intensity at sea level can be expressed from (28) as,

$$H(\Delta,\ 28.6)=K/\Delta^\gamma. \tag{29}$$

The value of the constant K is largely dependent on the absolute intensity of

primary rays, A. If we adopt

$$A=1.32\times10^{10}\ hr^{-1}m^{-2}, \tag{30}$$

following Hayakawa and Nishimura for $\lambda=1.8$[9], K turns out to be

$$K=1.19\times10^{3}\ hr^{-1}, \tag{31}$$

provided $k=1$. This value is twice as large as the most accurate experimental value of Cocconi and Cocconi Tongiorgi[13],

$$K=6.50\times10^{2}\ hr^{-1}. \tag{31'}$$

We do not consider the case $\lambda=2.0$, because this value of λ gives too steep density spectrum. If we adopt Heisenberg's spectrum[5], i.e. $A=1.15\times10^{9}$, however, this results in too small value $K=1.04\times10^{2}$. This discrepancy can not be improved even in the case of electron primary hypothesis, as mentioned by Williams[12], and it means that Heisenberg's spectrum is not correct.

Here, we have assumed the undetermined parameter k as unity in the charge exchange process. The probability for charge exchange, k, may be smaller than 1. If this is the case, we must seek the main mechanism for the production of the electronic-rays other than this process. For example, the decay photons from neutral mesons may be the main part of the electronic component. It is a very interesting problem to inquire about the cross-section for meson production and the life-time of the neutral meson, which are capable to explain the frequency of extensive air showers. We see from the above analysis, that the contribution from the neutral mesons should be the same order of magnitude as from the charge exchange process for $k=1$. Moreover, the probability for meson production must gradually increase with the increassing energy of an incident nucleon, since the contribution from the former process is slightly smaller

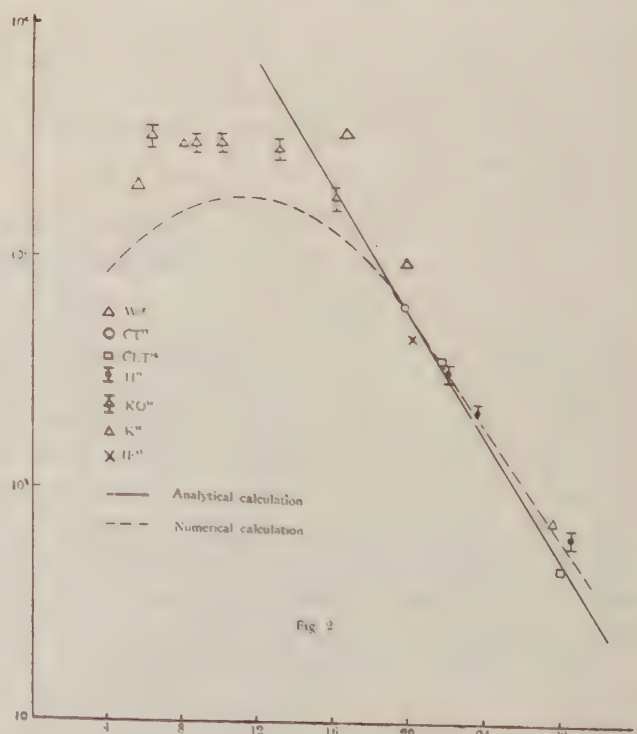

Fig. 2. Altitude variation of shower frequency.

than that from the latter in the moderate high energy region and cross-section for the latter process increases in propoition to Y_0. This will be realized by the fact that either the multiplicity of produced mesons increases, or the meson producing ability of recoil nucleons becomes effective, or both.

The altitude variation (28) is compared with experiments of Williams[12] by ionization chambers and of Cocconi *et all*[13), 14)], Hilberry[15)], Kraybill *et al*[16)], Ise and Fretter[17)] by counter trays (Fig. 2). We normalize the intensity at $l=20$ ($700\,g/cm^2$).

As is readily seen, the calculated variation is too steep, probably because of our rough approximation, especially in the interchange of the order of integrals and the extension of the integral region. Such a fault will be improved by numerical calculation in the next section.

§ 5. Numerical Calculation and Its Results.

The calculation in the preceding sections is convenient to see the general behaviour of extensive air showers, but not responsible to represent the detailed features such as the variation of γ with density and altitude and the altitude variation. We discuss qualitatively what sorts of modification will ta e place.

For showers with great density, the primary energy responsible to them will be very large and sometimes the showers will not yet reach their maxima at the observation station. Then the frequency for larger showers may be smaller than that expected by our approximate treatment. Such situation will be more effective for higher altitudes and will make γ larger for larger density.

In order to make sure of the above discussion and to get the good altitude dependence, we perform the numerical calculation. Firstly, formula (8) is integrated numerically. Secondly, it is solved with respect to Y_0 graphically. Lastly, the integrations over r and zenith angle are performed numerically. Thus we obtain $H(\Delta, l)$.

The density spectra at sea level ($l=28.0$) and at mountain elevation ($l=20$) are represented in Fig. 3, compared witn the experimental date of Cocconi *et al*[13)].

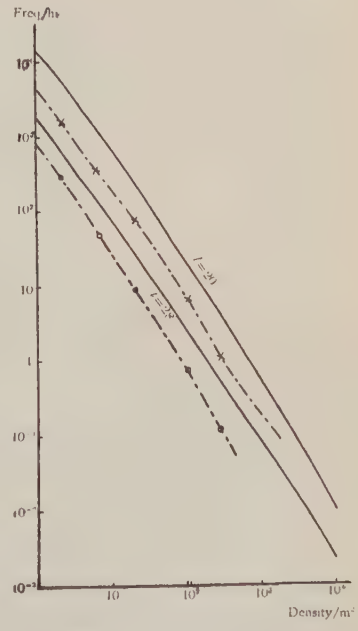

Fig. 3. Density spectrum; upper ——; theoretical, —·—·—: experimental at $l=28$, lower ——: theoretical, —·—·—: experimental at $l=20$,

The calculated density spectra are approximately represented in the neighbourhood of $\Delta=100\text{m}^{-2}$ by $K\Delta^{-\gamma}$ with

$K=2.43\times10^3\ hr^{-1}, \quad \gamma=1.51, \quad \text{for } l=28,$

$$K=2.18\times10^4\ hr^{-1},\qquad \gamma=1.59,\qquad \text{for } l=20, \tag{32}$$

for the charge exchange process, $k=1$. We see the value of γ gradually increases with altitude. The experimental values of Cocconi *et al*[13] are,

$$\begin{aligned} K&=7.20\times10^2\ hr^{-1}, &\gamma&=1.48, &&\text{for } l=28,\\ K&=6.50\times10^3\ hr^{-1}, &\gamma&=1.46, &&\text{for } l=20. \end{aligned} \tag{32'}$$

Considering the uncertainty in both the experiment and the numerical calculation, the agreement between them is said to be satisfactory, except a little faster increase of γ with altitude in our calculation than in the experimental value.

The variations of γ with density $\varDelta$ are given by,

$$\begin{aligned} \gamma&=1.35+0.08\ \log \varDelta, &&\text{for } l=28,\\ \gamma&=1.35+0.12\ \log \varDelta, &&\text{for } l=20. \end{aligned} \tag{33}$$

While, Cocconi *et al*[13] have given as,

$$\begin{aligned} \gamma&=1.31+0.086\ \log \varDelta, &&\text{for } l=28,\\ \gamma&=1.26+0.099\ \log \varDelta, &&\text{for } l=20. \end{aligned}$$

This variation of γ with $\varDelta$ explains the large value of γ for the larger density measured in the ionization chamber experiment of Williams[12] at $l\approx20$ and ≈16.

The altitude dependence of showers with $\varDelta=40\ \mathrm{m}^{-2}$, corresponding the measurement of Kraybill[16], is given in Fig. 2 by a dashed line, comparing with the experimental results, where we normalize them at $l=20$. The agreement with the experimental value is better for this case than for the analytical approximation, as expected in § 4. The discrepancy near the maximum will be due to our approximation as mentioned in the next section.

§ 6. Directional distribution.

Further knowledge about the structure of extensive air showers is obtained by the directional distribution. According to the electron primary hypothesis, the distribution is too broad[12]. Our calculation results in the considerable agreement with experimental one, as was shown in a previous note[18]. This result is not changed by the numerical calculation because larger zenith angle corresponds to deeper atmospheric depth. The calculated and the experimental one[19] are compared at $l=20$ in Fig. 4.

When we concern the directional distribution at $l\approx8$, the calculated result shows so extraordinary behaviour that the vertical intensity is smaller than the oblique one. This indicates that the vertical shower is too young and does not yet reach the maximum but the oblique shower develops more than the vertical one.

This result completely contradicts with the observation[20]. The reason for this discrepancy may be due to our wrong approximation that a primary proton falls

down through the atmosphere without multiplication. In reality, the energy of a primary proton will be imparted into many nucleons and nuclear mesons, by the collisions with air nuclei, each of which can produce electronic rays by a nuclear interaction or a spontaneous decay. Thus an air shower is composed of a number of such showers with smaller energies and reaches the maximum faster than our approximation. Such a modification is also necessary to treat the extensive air shower beyond its maximum. Accordingly, the agreement between theory and experiment in the altitude variation may not be conclusive. A further reason for the discrepancy in the directional distribution may lie in the use of Moliere function for younger shower, since Moliere function is grounded theoretically only for shower maximum, though confirmed experimentally in wider range. If there were an accurate treatment of extensive air showers in higher altitudes accounting for above consideration, one could definitely know their nature.

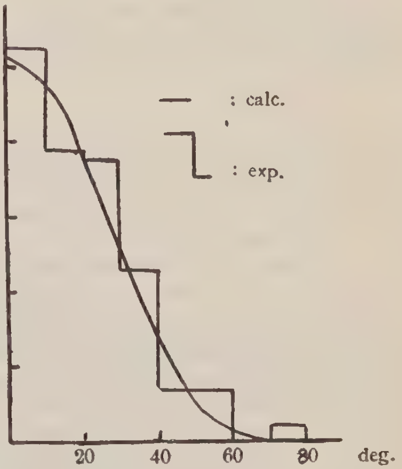

Fig. 4. Directional distribution of showers at $l=20$; full curve: theoretical, block diagram: experimental.

In spite of such unsatisfactory points, we may say that the proton primary hypothesis is superior to the electron primary one[21], especially in the altitude and directional variation.

§ 7. Conclusion and Summary.

The behaviour of extensive air showers is explained better by proton primary hypothesis. The density spectrum reflects the primary spectrum, $AE^{-\lambda}$, $\lambda = 1.8$. The cross-section for the production of electronic rays is explained by the decay of neutral mesons or the charge exchange process. If it is the former case, the cross-section for meson production will slowly increase with energy, and an indefinite constant k in the charge exchange will be smaller than unity. Of course, it is possible that both processes exist together. The altitude variation in lower altitude is mainly affected by the absorption of parent nucleons. Near the shower maximum, the approximation of the absorption of a single nucleon becomes wrong and it is necessary to consider the multiplication process of the nucleon component.

In spite of some faults as above, we may say that (1) the primary spectrum can be extrapolated from lower energy regions, (2) electronic rays are produced by nuclear interaction with a considerable cross-section, and (3) the puentum electrodynamics for electrons and photons is still valid for such great energy as appeared in extensive air showers.

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On the Convergence of the Perturbation Method, II. 2.

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§ 4. Higher Approximations.

If we further assume the existence of $VSV\varphi_0$, the six coefficients $\lambda^{(1)}$, $\lambda^{(2)}$, $\lambda^{(3)}$, $\lambda^{(4)}$, $\varphi^{(1)}$, and $\varphi^{(2)}$ are certainly significant. In this case the following relations hold :

$$\left.\begin{aligned}\lambda_\varkappa&=\lambda_0+\varkappa\lambda^{(1)}+\cdots\cdots+\varkappa^4\lambda^{(4)}+o(\varkappa^4),\\ \varphi_\varkappa&=\varphi_0+\varkappa\varphi^{(1)}+\varkappa^2\varphi^{(2)}+o(\varkappa^2).\end{aligned}\right\}\tag{21}$$

The proof of (21) is analogous to the case of (19). Again we have to content ourselves with a partial proof. Take as an approximate eigen-vector of $H_\varkappa$

$$w_\varkappa=(1+\varkappa^2\,\|\,\varphi^{(1)}\,\|\,^2)^{-\frac{1}{2}}(\varphi_0+\varkappa\varphi^{(1)}),\quad (\|\,w_\varkappa\,\|=1).\tag{22}$$

Then we have by direct calculation

$$\left.\begin{aligned}\eta_\varkappa&=(H_\varkappa w_\varkappa,\ w_\varkappa)\\ &=\lambda_0+\varkappa\lambda^{(1)}+\varkappa^2\lambda^{(2)}+\varkappa^3\lambda^{(3)}+o(\varkappa^4),\\ \varepsilon_\varkappa&=\|\,(H_\varkappa-\eta_\varkappa)w_\varkappa\,\|=o(\varkappa^2).\end{aligned}\right\}\tag{23}$$

Hence we have, according to the theorem of E. referred to,

$$\lambda_\varkappa=\lambda_0+\varkappa\lambda^{(1)}+\varkappa^2\lambda^{(2)}+\varkappa^3\lambda^{(3)}+o(\varkappa^4),$$

$$\varphi_\varkappa=\varphi_0+\varkappa\varphi^{(1)}+o(\varkappa^2),$$

which are slightly less precise than (21). As before (21) are proved by taking better approximate eigen-vector than (22).

These results are generalized to higher approximations as follows. Suppose that the quantities

$$VS^{r_1}VS^{r_2}V\cdots VS^{r_k}V\varphi_0$$

$$(k\leqq n-1,\ r_i\geqq 1,\ r_1+\cdots+r_k\leqq n-1)$$

containing at most n V's are all existent. Then the coefficients of (13) and (14) are seen to be significant up to the term $\varkappa^{2n}$ and $\varkappa^n$ respectively. It can also be shown that

$$(H_\varkappa-\lambda_0-\varkappa\lambda^{(1)}-\cdots-\varkappa^{n-1}\lambda^{(n-1)})\cdot(\varphi_0+\varkappa\varphi^{(1)}+\cdots+\varkappa^{n-1}\varphi^{(n-1)})=o(\varkappa^n), \tag{24}$$

the left-hand side being certainly significant. In fact, the coefficients $\lambda^{(n)}$ and $\varphi^{(n)}$ are formally determined just in such a way that (24) holds for each n. Hence if we apply the theorem of *E.*, on taking

$$w_\varkappa=\|\psi_\varkappa\|^{-1}\psi_\varkappa, \ (\psi_\varkappa=\varphi_0+\varkappa\varphi^{(1)}+\cdots+\varkappa^{n-1}\varphi^{(n-1)})$$

as an approximate eigen-vector of $H_\varkappa$, we obtain easily

$$\varepsilon_\varkappa=\|(H_\varkappa-\eta_\varkappa)w_\varkappa\|=o(\varkappa^n), \quad \eta_\varkappa=(H_\varkappa w_\varkappa, w_\varkappa).$$

As before it follows that

$$|\lambda_\varkappa-\eta_\varkappa|=o(\varkappa^{2n}), \ \|\varphi_\varkappa-w_\varkappa\|=o(\varkappa^n).$$

Clearly $\eta_\varkappa$ must coincide with (13) up to the order $\varkappa^{2n-1}$. Also $w_\varkappa$ and $\psi_\varkappa$ coincide as far as $\varkappa^{n-1}$. Thus we obtain

$$\lambda_\varkappa=\lambda_0+\varkappa\lambda^{(1)}+\cdots+\varkappa^{2n-1}\lambda^{(2n-1)}+o(\varkappa^{2n}),$$

$$\varphi_\varkappa=\varphi_0+\varkappa\varphi^{(1)}+\cdots+\varkappa^{n-1}\varphi^{(n-1)}+o(\varkappa^n).$$

More detailed argument shows, as before, that the last terms $o(\varkappa^{2n})$ and $o(\varkappa^n)$ can be replaced by

$$\varkappa^{2n}\lambda^{(2n)}+o(\varkappa^{2n}), \quad \varkappa^n\varphi^{(n)}+o(\varkappa^n)$$

respectively. *Thus p.m. is in this case correct to 2n–th approximation for the eigen-value and to n–th approximation for the eigen-vector.*

§ 5. Degenerate Eigenvalues.

If the eigen-value λ_0 is degenerate ($m>1$), our argument becomes slightly complicated, although the main feature remains unaltered. We shall therefore restrict ourselves to the first and second approximations of eigen-values, which are the most important cases in application. The conditions i), ii) or i′), ii) are again assumed.

Let $\varphi_{01},\cdots\cdots,\varphi_{0m}$ be the ortho-normal system composed of the eigen-vectors of H_0 associated with the eigen-value λ_0. We assume that they all belong to the domain of V. Then we can construct the Hermitian matrix

$$(V\varphi_{0i}, \varphi_{0j}), \quad (i, j=1, \cdots, m). \tag{25}$$

Further we assume, for simplicity's sake, that this matrix has distinct eigen-values

$$\lambda_1^{(1)}<\lambda_2^{(1)}<\cdots<\lambda_m^{(1)}. \tag{26}$$

We may assume, without loss of generality, that the matrix (25) is already diagonalized. Thus we have

$$(V\varphi_{0i}, \varphi_{0j})=\delta_{ij}\lambda_i^{(1)}, \ (\varphi_{0i}, \varphi_{0j})=\delta_{ij}. \tag{27}$$

Then we apply the result of E, § 3. For this purpose, we take $\varphi_{01}\cdots$, φ_{0m} as approximate eigen-vectors of $H_\varkappa$. By virtue of (27) we have

$$(H_\varkappa\varphi_{0i}, \varphi_{0j})=\delta_{ij}\eta_{\varkappa i}, \quad \eta_{\varkappa i}=\lambda_0+\varkappa\lambda_i^{(1)}, \tag{28}$$

and

$$\varepsilon_{\varkappa i}=\| (H_\varkappa-\eta_{\varkappa i})\varphi_{0i} \| =\varkappa \| (V-\lambda_i^{(1)})\varphi_{0i} \| . \tag{29}$$

According to eqs. (30) and (33) of $E.$, we have

$$-\sum_{j=i}^{m}\frac{\varepsilon_{\varkappa j}^2}{\beta-\eta_{\varkappa j}}\leq\lambda_{\varkappa i}-\eta_{\varkappa i}\leq\sum_{j=1}^{i}\frac{\varepsilon_{\varkappa j}^2}{\eta_{\varkappa j}-\alpha}.$$

Since all quantities $\eta_{\varkappa i}-\alpha$, $\beta-\eta_{\varkappa i}$ have a positive lower bound d independent of $\varkappa$, we obtain

$$\lambda_{\varkappa i}=\lambda_0+\varkappa\lambda_i^{(1)}+o(\varkappa^2). \tag{30}$$

Next we turn to the estimation of the eigen-vectors. Let $\varphi_{\varkappa i}$ be the correct eigen-vectors of $H_\varkappa$ associated with the eigen-values $\lambda_{\varkappa i}(i=1, \cdots, m)$. $\varphi_{\varkappa i}$ are determined except for a phase factor since $\lambda_{\varkappa i}$ are different from each other by (26) and (30). Noting that $(H_\varkappa-\lambda_{\varkappa i})\varphi_{\varkappa i}=0$, we have

$$\begin{aligned}0&=((H_\varkappa-\lambda_{\varkappa i})\varphi_{\varkappa i}, \varphi_{0j})\\&=(\varphi_{\varkappa i}, (H_0+\varkappa V-\lambda_{\varkappa i})\varphi_{0j})\\&=(\varphi_{\varkappa i}, (\lambda_0+\varkappa\lambda_j^{(1)}-\lambda_{\varkappa i})\varphi_{0j}+\varkappa(1-E_0)(V-\lambda_j^{(1)})\varphi_{0j}),\end{aligned}$$

where we used the relation

$$E_0(V-\lambda_j^{(1)})\varphi_{0j}=0$$

which is a direct consequence of (27), E_0 being the projection on the eigen-space of H_0 associated with the eigen-value λ_0 (see (6)). On putting

$$c_{ij}(\varkappa)=(\varphi_{\varkappa i}, \varphi_{0j}), \ (i, j=1, \cdots, m),$$

we obtain

$$(\lambda_{\varkappa i}-\lambda_0-\varkappa\lambda_j^{(1)})c_{ij}(\varkappa)=\varkappa((1-E_0)\varphi_{\varkappa i}, (V-\lambda_j^{(1)})\varphi_{0j}). \tag{31}$$

On the other hand, it can be shown from (29) that

$$(1-E_0)\varphi_{\varkappa i}=o(\varkappa). \tag{32}$$

This is a specialization of (7) under the present assumption that $V\varphi_{0i}(i=1, \cdots, m)$ exist. From (31), (32) and (30) we conclude

$$c_{ij}(\varkappa)=o(\varkappa) \quad \text{for } i\neq j$$

and, combining again with (32), we obtain

$$\varphi_{\varkappa i}=\varphi_{0i}+o(\varkappa), \tag{33}$$

provided $\varphi_{\kappa i}$ are properly normalized.

On using better approximate eigen-vectors, we can improve the result as before and show, under the same assumptions, that

$$\left.\begin{aligned}\lambda_{\kappa i}&=\lambda_0+\kappa\lambda_i^{(1)}+\kappa^2\lambda_i^{(2)}+o(\kappa^2)\ ,\\ \varphi_{\kappa i}&=\varphi_{0i}+\kappa\varphi_i^{(1)}+o(\kappa)\ ,\end{aligned}\right\}\tag{34}$$

$$\lambda_i^{(2)}=-(SV\varphi_{0i},\ V\varphi_{0i})\ ,$$

$$\varphi_i^{(1)}=-SV\varphi_{0i}+S_i(VE_0)^{\dagger}SV\varphi_{0i}\ .$$

Here S is defined as before by (17) ; S_i is given by

$$S_i=\sum_{j\neq i}(\lambda_j^{(1)}-\lambda_i^{(1)})^{-1}P_{[\varphi_{0j}]}\ ,$$

$P_{[\varphi]}$ being the projection operator on the one-dimensional space $[\varphi]$; E_0 is used above and

$$E_0=\sum P_{[\varphi_{0i}]}$$

holds ; finally $(VE_0)^{\dagger}$ denotes the adjoint of the operator VE_0 which is shown to be *bounded.* Of course (34) are in agreement with usual formulas.

Thus p.m. is valid to the 2nd approximation with respect to the eigen-values and to the 1st approximation with respect to the eigen-vectors provided $V\varphi_{0i}(i=1,\cdots,m)$ exist.

Higher approximations can be treated in similar manner and lead to results corresponding to those of § 4. But it will not be necessary to state them in detail.

§ 6. Conclusions and Examples.

1) Our results can be summarized as follows : *roughly speaking, p.m. gives asymptotic series in ascending power of* κ *which are correct so far as the coefficients can be calculated by means of operations within the Hilbert space.* It is important to note that this was established independently of the convergence or divergence of the formal series. In fact, it is rather usual that the series have only finite significant terms.

On the other hand, p.m. can often lead to fallacious results if the restriction just stated is violated In some cases the mishap is revealed by the occurrence of the so-called divergence difficulty (some coefficient becomes infinity), but this is not always the case and one is exposed to the danger of obtaining spurious results. Also the occurrence of the divergence difficulty does not necessarily mean that the eigen-value is really infinite or non-existent. It simply indicates that p.m. is not applicable to that order of approximation.

As an example, consider the eigen-value problem relating to the vibration of a uniform string with small rigidity and with both ends clamped. On choosing appropriate units, the differential equation is given by

$$\frac{d^2\varphi}{dx^2}-\varkappa\frac{d^4\varphi}{dx^4}+\lambda\varphi=0,\ (0\leq x\leq 1) \tag{35}$$

with boundary conditions (abr. b.c.)

$$\varphi(0)=\varphi'(0)=\varphi(1)=\varphi'(1)=0. \tag{36}$$

In order to apply p.m. we have to put

$$H_0=-d^2/dx^2 \quad \text{with b.c.} \quad \varphi(0)=\varphi(1)=0, \tag{37}$$

$$V=d^4/dx^4 \qquad \text{with b.c. (36).}$$

It is easily seen that the operator $H_0+\varkappa V$ with b.c. (36) is essentially self-adjoint[7] if $\varkappa>0$, but this is not the case for $\varkappa=0$, for b.c. (36) is too strong for the 2nd order differential operator $-d^2/dx^2$. Thus the condition i) of § 2 is not satisfied. But both H_0 and V are positive definite, so that the Friedrichs extension (see § 2) of $H_0+\varkappa V$ is defined and shown to coincide for $\varkappa=0$ with H_0 with just the above b.c. (37). This is the *mathematical* reason why the particular b.c. (37) should be chosen for H_0, although this is rather evident from the physical intuition. Thus the condition i') of § 2 is satisfied. Also H_0 has pure point spectrum consisting of eigenvalues

$$\lambda_n=n^2\pi^2,\ (n=1,\ 2,\cdots\cdots),$$

so that the condition ii) is also satisfied .(see the case B) of § 2). It follows from § 2 that p.m. is valid at least to the 0-th approximation, i e. the eigen-values and eigen-functions of (35) are continuous with respect to $\varkappa$ at $\varkappa=0$.

But higher appıoximations than the 0-th order are not valid. The eigen-functions of H_0 are given by

$$\varphi_n=\sqrt{2}\ \sin n\pi x,\ (n=1,\ 2,\ \cdots\cdots).$$

Since these functions do not satisfy b.c. (36), the operator V cannot be operated on φ_n. Hence $V\varphi_n$ are not existent, and our conditions for higher approximations of p.m. are not satisfied.

In fact, the n-th eigen-value $\lambda_n(\varkappa)$ of (35) has the form

$$\lambda_n(\varkappa)=n^2\pi^2+4n^2\pi^2\varkappa^{\frac{1}{2}}+o(\varkappa),$$

as was shown by Rayleigh[14]. Thus it is clear that p.m. is in this case applicable only to the 0-th approximation.

Nevertheless, if we applied the formulas of p.m. quite formally, ignoring the fact that $V\varphi_n$ does not exist and operating the differential operator $V=d^4/dx^4$ formally on φ_n, we should obtain finite coefficients of $\varkappa$, $\varkappa^2$ etc. in the formal series. But this series is, of course, meaningless. This example will be sufficient to show that our conditions are by no means superfluous.

2) As we stated in § 2, the assumptions i), ii) or i'), ii) are not very restrictive. But as ii) is rather complicated, it is desirable to replace it by weaker condition.

Now it is possible that, although ii) is not satisfied when $H_\varkappa$ is considered in the whole Hilbert space, it is satisfied in each of invariant subspaces of the latter ("invariant" means that they reduce[15] the operators H, and V). In such a case, our theory can be applied in each subspace separately, and the applicability range of p.m. is considerably extended.

As a simple example, we shall refer to the Hamiltonian operator of the Zeeman effect of the hydrogenic atom[16]. Here these invariant subspaces are those in which the magnetic quantum number takes on definite values.

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6) Reference (3), p. 33.
7) Reference (3), p. 51.
8) Reference (3), p. 45.
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15) Reference (3), p. 150.
16) Cf. F. Rellich: Math. Ann. **116** (1939), 555.

Note added in proof: The reader will have noticed that there has been a gap in our system of sufficient conditions given in § 3–§ 5. Namely all of them had in common the property that they give the expansion of eigenvalues up to *even* orders of approximation, and there was no theorem giving an expansion up to an odd order.

Recently we have succeeded in obtaining such theorems. For instance in the non-degenerate case, the expansion of the eigenvalue is shown to be valid to the first approximation if, under other conditions of general character, the *formal* expression $\lambda^{(1)}=(V\varphi_0, \varphi_0)$ is finite when interpreted e.g. as an integral in the configuration space (even if $V\varphi_0$ itself does not exist). With these supplements, our system of conditions may be considered as complete.

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Note on the Theory of the Frequency Spectrum of Crystalline Solid.

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Resumé. In order to examine the accurancy of Houston's method of approximation for calculating the frequency spectrum of crystalline solid, the approximate frequency spectrum of square lattice obtained from this method is compared with the exact one obtained from Montroll's method. In the neighborhood of the origin the curve obtained from Houston's approximation agrees rather exactly with the accurate curve, while its slope becomes too steep as the frequency approaches the first peak. The portion of this approximate curve to the right of the first peak has no physical significance. The frequency distribution function consists of two or more analytically different parts. If we try to apply Houston's method to such parts separately, the frequency distribution function without spurious peaks will be obtained. And it is of importance to investigate the topological properties of the curve of equal frequency in Brillouin zone for the purpose of carrying out the calculation according to this method.

§ **1.** In recent years, there appeared three illuminating theories of the frequency spectrum of crystalline solid based on the Born-von Kárman model, two by Montroll and one by Houston. First, Montroll[1] reduced the calculation of the frequency distribution function to the so called moment problem in statistics, where moments are given by the trace of the products of the matrix in the eigenwert problem as H. Thirring[2] already found. This method of approximation for the frequency distribution function, hereinafter referred to as Thirring-Montroll's method or tracemoment method, gives the mean approximation in the sense of the least square method, but not the local approximation. And calculations of moments of higher order necessary to obtain the more accurate distribution funntion are tedious. Secondly, Montroll[3] gave the exact solution for the square lattice, according to which the distribution function has two points of logarithmic infinity. But, it seems to be difficult to extend Montroll's exact solution to three dimensional lattices, although it is desirable. Hereinafter Montroll's paper concerning his exact solution is referred to as M2. Finally, Houston[4] developed one approximate method, where unknown solutions of the secular equation are interpolated from known ones of the particular directions in Brillouin zone along which we easily obtain solution of the secular equation. This method of approximation, hereinafter referred to as Houston's method, is easy in calculation, but has unfortunately spurious peaks. In spite of its defects, it seems to have some convenience because of its simplicity. Therefore, we consider it is worth while to inquire the nature of Houston's method of approximation.

S. Huzinaga[5] investigated the validity of Houston's method of approximation for the case of simple cubic lattice in the limit of $\tau=0$, where the interaction between second nearest neighbors vanishes, obtaining the conclusion that Houston's approximation does not give a good result in this case.

We shall here discuss Houston's method of approximation with regard to the square lattice in the case of $\tau=1/3$, where the interaction between the first nearest neighbors is four times as that between second nearest neighbors, because the result obtained from this case can be compared with Montroll's exact solution. Thus, we shall have some informations about the nature of Houston's method of approximation in three dimensional cases by analogy with that of two dimensional ones.

An attempt is made to salvage Houston's method of approximation for the square lattice, and the frequency distribution function obtained from this attempt is in good agreement with Montroll's exact solution. It is desirable to extend it to three dimensional lattices, such as simple cubic, body-centered cubic and face centered cubic lattice, and to compare them with the distribution function of simple cubic lattice calculated by Blackman[6] and Montroll,[1] of body-centered by Fine[7] and Montroll,[1] and of face-centered by Leighton[8] respectively.

In the last section, we shall consider the usefulness of Houston's method of approximation in connection with the thermodynamical quantities, in contrast with the usefulness of tracemoment method.

§ II. Let us consider the square lattice consisting of N^2 atoms of mass M, and satisfying the Born-von Kármán's periodicity condition.

The frequency spectrum $\{\nu\}$ of the lattice vibrations of the square lattice is determined by the secular equation

$$\begin{vmatrix} A(\varphi_1,\ \varphi_2)-4\pi^2\nu^2 M & B(\varphi_1,\ \varphi_2) \\ B(\varphi_1,\ \varphi_2) & A(\varphi_2,\ \varphi_1)-4\pi^2\nu^2 M \end{vmatrix}=0, \tag{1}$$

where

$$A(\varphi_i,\ \varphi_j)=2\alpha(1-\cos\varphi_i)+4\gamma(1-\cos\varphi_i\cos\varphi_j),$$

$$B(\varphi_1,\ \varphi_2)=4\gamma\sin\varphi_1\sin\varphi_2\ ,$$

when φ_1 and φ_2 are the coordinates of the reciprocal lattice point varying from 0 to 2π, and α is the restoring force constant between first nearest neighbors, γ the one between second nearest neighbors.

Now, we transform the rectangular coordinates $(\varphi_1,\ \varphi_2)$ to the polar coordinates $(r,\ \vartheta)$ determined by

$$\varphi_1=r\cos\vartheta,\quad \varphi_2=r\sin\vartheta\,.$$

Then, solving Eq. (1) with regard to r, we obtain

$$r=\chi(\vartheta,\ \nu) \tag{2}$$

which represents a curve in the reciprocal lattice space, if ν is kept constant. We call it the curve of equal frequency.

The secular equation (1) and accordingly the curve of equal frequency remain invariant under the rotation of $\pi/2$ and the transformation : $\vartheta \to -\vartheta$. Hence, Eq. (2) can be expanded in the following Fourier series :

$$r=\chi(\vartheta, \nu) = \frac{1}{2}a_0(\nu)+\sum_{n=1}^{\infty}a_n(\nu)\cos 4n\vartheta\,. \tag{3}$$

Let $A(\nu_1)$ be the area circumscribed by the φ_1- and φ_2-axes and the curve of equal frequency for $\nu=\nu_1$, then we have

$$A(\nu_1)=\frac{1}{2}\int_0^{\pi/2}[\chi(\vartheta, \nu_1)]^2 d\vartheta\,.$$

Substituting the expansion formula (3) into this expression and making use of the orthogonality relation, we obtain

$$A(\nu_1)=\frac{\pi}{4}\left[\frac{1}{4}a_0(\nu_1)^2+\sum_{n=1}^{\infty}a_n(\nu)^2\right]. \tag{4}$$

The frequency distribution function $f(\nu)$ is, then, obtained by the differentiation of $A(\nu)$ with respect to ν:

$$f(\nu)=\frac{1}{2\pi}\left[\frac{1}{4}a_0(\nu)\frac{da_0(\nu)}{d\nu}+\sum_{n=1}^{\infty}a_n(\nu)\frac{da_n(\nu)}{d\nu}\right],$$

where Eq. (4) is devided by the square of π in view of the normalization of $f(\nu)$. To obtain $f(\nu)$, we must calculate $\{a_n(\nu)\}$ which are formally expressed by

$$a_n(\nu)=\frac{4}{\pi}\int_0^{\pi/2}\chi(\vartheta, \nu)\cos 4n\vartheta d\vartheta\,. \tag{5}$$

To determine these Fourier coefficients, according to the similar consideration as used in the Houston's theory, we solve the secular equation (1) along the lines $\varphi_2=0$ and $\varphi_1=\varphi_2$, and obtain $a_n(\nu)$'s from these solutions by making use of Eq. (3), where $a_n(\nu)$'s are retained only up to the second. Thus we have

$$\begin{cases} a_0(\nu)=\chi(0, \nu)+\chi(\pi/4, \nu) \\ a_1(\nu)=\frac{1}{2}\{\chi(0, \nu)-\chi(\pi/4, \nu)\}, \end{cases} \tag{6}$$

where $\chi(0, \nu)$ and $\chi(\pi/4, \nu)$ are respectively the solutions of Eq. (1) along the directions of $\varphi_2=0$ and $\varphi_1=\varphi_2$, Substitution of these equations into Eq. (3) leads to the approximate expression of the curve of equal frequency. Similarly, the distribution function $f(\nu)$ can be written in the form

$$f(\nu)=(4\pi)^{-1}\left[\chi(0, \nu)\frac{d\chi(0, \nu)}{d\nu}+\chi(\pi/4, \nu)\frac{d\chi(\pi/4, \nu)}{d\nu}\right].$$

Since the secular equation (1) has two real positive roots, there exist two frequencies which correspond to one point in the reciprocal lattice space. Then, for the convenience' sake, we consider two different lattice spaces on the analogy of the Riemann surface in the theory of complex function, so that every point of each space corresponds to one and only one frequency. Then, there exists only one curve of equal frequency in each space for the definite frequency. We call one whose spectrum extends over large frequency region the first branch, and the other the second branch. Taking the mean value of the distribution functions for both branches, we shall obtain the distribution function from Houston's standpoint.

Houston expanded the distribution function in spherical harmonics directly, but our formulation is more essential, and it seems to be advantageous to criticize Houston's method.

§ **III.** Let us introduce, after Montroll, two parameters, τ and s, defined by

$$2\gamma/a=(1-\tau)^{-1}\tau, \quad \text{and} \quad s=\pi\{M(1-\tau)^{1/2}/a\}^{1/2}\nu, \quad \text{or} \quad s=\nu/\nu_L,$$

ν_L being the maximum frequency, so that s satisfies the inequality $0\leq s\leq 1$. The solutions of the secular equation (1) along the directions $\varphi_2=0$ and $\varphi_1=\varphi_2$ are respectively written in the following forms.

1st branch:

$$\chi_1(0,s): \qquad r=2\sin^{-1}s, \tag{8}$$

$$\chi_1(\pi/4,s); \quad r=2\sqrt{2}\,\sin^{-1}\left[\left(\frac{3\tau+1}{8\tau}\right)^{1/2}(1-Q)^{1/2}\right], \tag{9}$$

where $$Q=[1-16\tau(3\tau+1)^{-2}s^2]^{1/2};$$

2nd branch;

$$\chi_2(0,s): \qquad r=2\sin^{-1}[\tau^{-1/2}s], \tag{10}$$

$$\chi_2(\pi/4,s): \quad r=2\sqrt{2}\,\sin^{-1}[(1-\tau)^{-1/2}s], \tag{11}$$

where we use the relation: $r=\sqrt{2}\,\varphi$ along the direction $\varphi_1=\varphi_2(\equiv\varphi)$.

If we set $\tau=1/3$, and calculate $a_n(s)$'s for $n=0, 1$, the following expressions are obtained:

(a) 1st branch:

$$a_0(s)=2\sin^{-1}s+2\sqrt{2}\,\sin^{-1}\left[\frac{\sqrt{3}}{2}(1-Q)^{1/2}\right],$$

$$a_1(s)=\ \sin^{-1}s-\ \sqrt{2}\,\sin^{-1}\left[\frac{\sqrt{3}}{2}(1-Q)^{1/2}\right],$$

where $$Q=(1-4s^2/3)^{1/2}$$

Then, the curve of equal frequency can be expressed by

$$
\begin{aligned}
r&=\chi_1(\vartheta, s)\\
&=\sin^{-1}s+\sqrt{2}\,\sin^{-1}\left[\frac{\sqrt{3}}{2}(1-Q)^{1/2}\right]\\
&\quad+\left\{\sin^{-1}s-\sqrt{2}\,\sin^{-1}\left[\frac{\sqrt{3}}{2}(1-Q)^{1/2}\right]\right\}\cos 4\vartheta,
\end{aligned}
\tag{12}
$$

and accordingly the distribution function in the first branch $f_1(\nu)$ is given by

$$
\nu_L f_1(s)=\pi^{-1}\left[\frac{\sin^{-1}s}{(1-s^2)^{1/2}}+\frac{3}{4}\,\frac{\sin^{-1}\left[\frac{\sqrt{3}}{2}(1-Q)^{1/2}\right]}{(Q+1/3)^{1/2}(1-Q)^{1/2}Q}s\right]. \tag{13}
$$

(*b*) 2nd branch:

$$
\begin{aligned}
r&=\chi_2(\vartheta, s)\\
&=\sin^{-1}\sqrt{3}\,s+\sqrt{2}\,\sin^{-1}\frac{\sqrt{3}}{2}s+\{\sin^{-1}\sqrt{3}\,s-\sqrt{2}\sin^{-1}\sqrt{3/2}\,s\}\cos 4\vartheta,
\end{aligned}
\tag{14}
$$

and

$$
\nu_L f_2(s)=\pi^{-1}\left[\sqrt{3}\,\frac{\sin^{-1}\sqrt{3}\,s}{(1-3s^2)^{1/2}}+\sqrt{6}\,\frac{\sin^{-1}\sqrt{3/2}\,s}{(1-3s^2/2)^{1/2}}\right]. \tag{15}
$$

We then have the distribution function averaged over both branches, namely

$$
f(\nu)=\frac{1}{2}(f_1(\nu)+f_2(\nu))\,.
$$

This is the distribution function deduced by Houston's method, and is plotted in Fig. 1. By comparison of this figure with Fig. 1 in M2, we see that the first and third peaks are in accordance with exact one, but the second and fourth ones are spurious. And that, even if we smooth out these spurious peaks moderately, the real distribution function cannot be reproduced by this distribution function. Actually, on the right side of the first peak the density of frequencies is poor as compared with the exact solution.

We can clearly understand these circumstances, if we plot the distribution functions $f_1(s)$ and $f_2(s)$ separately (cf. Fig. 2). As compared with Figs. 6 and 7 in M2, the distribution function of each branch reproduces the exact one to some extent up to the first peak, but the rest portion of each distribution function is very far from the exact one.

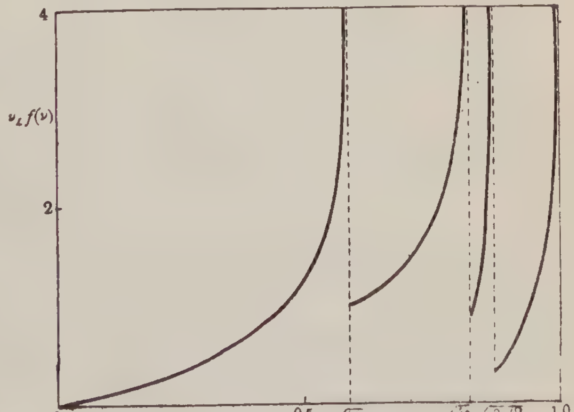

Fig. 1. The distribution function calculated by Houston's method ($\tau=1/3$).

Next, we will investigate the curve of equal frequency which are computed by Eqs. (12) and (14). These are plotted in Figs. 3 and 4. Fig. 3 shows a family of curves of equal frequency in the first branch, and should be compared with Fig. 5 in M2. Although the nearer the frequencies approach $s=\sqrt{3/2}$ the approximations of curves of equal frequency become the less accurate, they reproduce the actual situation to a certain degree of approximation. But, after crossing $s=\sqrt{3/2}$, it is impossible to plot a curve of equal frequency from the adjustment along these two directions, because there exists no point belonging to the first Brillouin zone beyond $\varphi_1=\pi$. In spite of this circumstance, according to Eq. (13), the distribution function has the remnant due to the first term in the right hand side of Eq. (13) beyond $s=\sqrt{3/2}$. It is equivalent to the rate of change of areas circumscribed by the dotted curve in Fig. 3. But these curves of equal frequency are entirely illusory, and so the distribution function loses the physical significance beyond $s=\sqrt{3/2}$. In the second branch, these circumstances are invariable, and the distribution function is transformed into a meaningless one beyond $s=1/\sqrt{3}$ at which the first peak occurs (cf. Fig. 4). In Fig. 4, the fictitious curves of equal frequency which extend along the direction $\varphi_1=\varphi_2$ have been omitted.

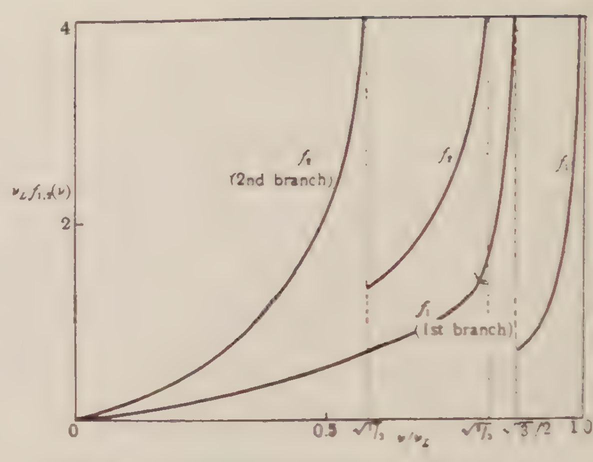

Fig. 2. The distribution function in each branch calculated by Houston's method. ($\tau=1/3$).

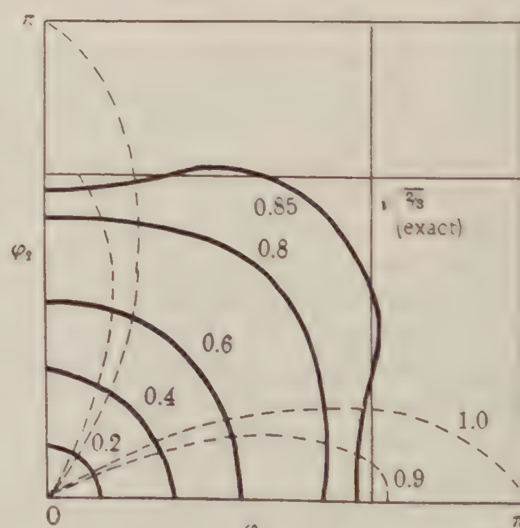

Fig. 3. The curves of equal frequency in the first branch computed from Eq. (12). The dotted curves are fictitious ones.

§**IV.** From the discussion made above, we have known that the distribution function of each branch by Houston's method is entirely fictitious beyond the first peak, but reproduces the actual circumstance to a certain degree of approximation up to the first peak.

Let us inquire the accuracy of Houston's approximation. On the one hand,

expanding Eqs. (13) and (14) in a power series of s, we have

$$\nu_L f_1(s) = 2\pi^{-1}s\left(1+\frac{11}{6}s^2 + \frac{593}{480}s^4 + \cdots\right), \qquad (16a)$$

$$\nu_L f_2(s) = 6\pi^{-1}s\left(1+\frac{3}{2}s^2 + 3\,s^4 + \cdots\right). \qquad (16b)$$

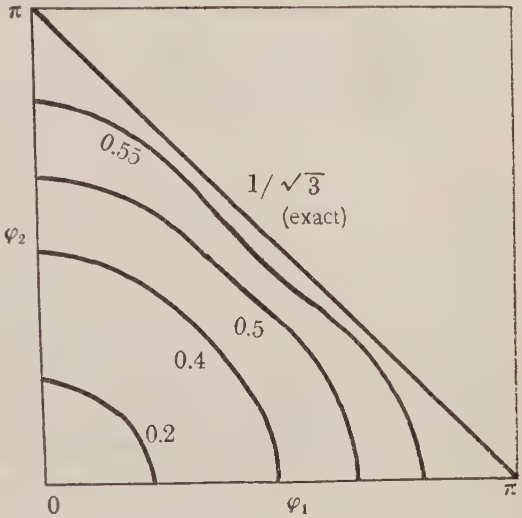

Fig. 4. The curves of equal frequency in the second branch computed from Eq. (14).

On the other hand, according to Montroll's exact solution, we have

$$\nu_L f_1(s) = 2\pi^{-1}s\left(1+\frac{5}{6}s^2 + \frac{117}{144}s^4 + \cdots\right), \qquad \text{(36) in M2.}$$

$$\nu_L f_2(s) = 6\pi^{-1}s\left(1+\frac{3}{2}s^2 + \frac{45}{16}s^4 + \cdots\right). \qquad \text{(33) in M2.}$$

Hence, for the first branch, $f_1(s)$ by Houston's approximation is correct up to the s term, and for the second branch $f_2(s)$ up to s^3 term.

We can then see that Houston's approximation is rather good in the neighborhood of the origin as Houston supposed. However, as τ decreases from 1/2 to 0, the first peak approaches the origin, and hence the valid interval of Houston's approximation decreases. (cf. V). But, as this decrease is as rapid as the square root of τ, it will be nòt so sensitive to the value of τ. The case $\tau=0$ discussed by S. Huzinaga is the worst one for Houston's method of approximation.

§ **V.** The point at which the distribution function has an infinity is given by the singular point of the expression (2) of the curves of equal frequency defined in the neighborhood of the origin. Beyond this point, the expression of curves of equal frequency becomes different from that. Probably, this singular point corresponds to the frequency at which the point belonging to the same frequency vanishes along either direction $\varphi_2=0$ or $\varphi_1=\varphi_2$. It is supposed that the condition described above depends upon the unique property which the directions $\varphi_2=0$ and $\varphi_1=\varphi_2$ possess. If we grant the above conjecture, it is possible to find the point of infinity immediately from Eqs. (8), (9), (10) and (11).

In the first branch, Eq. (8) does not confine s because of the inequality $0\leq s\leq 1$, but Eq. (9) confine s to the domain, in which following two conditions must be satisfied:

$$0\leq\left(\frac{3\tau+1}{8\tau}\right)^{1/2}(1-Q)^{1/2}\leq 1, \qquad (17)$$

and

$$Q \text{ is real, i.e.,} \quad 1-16\tau(3\tau+1)^{-2}s^2 \geq 0\,. \tag{18}$$

We shall consider two different cases:

(i) $\tau > 1/5$. In this case, Eq. (18) is automatically satisfied. So, s is confined to the domain $s \leq (3\tau+1)/4\tau^{1/2}$. Hence, the singular point is given by

$$s = (3\tau+1)/4\tau^{1/2}.$$

(ii) $\tau < 1/5$. The limitation of s due to Eq. (18) is given by

$$s \leq (1-\tau)^{1/2},$$

and it is more stringent condition than that of Eq. (17). Hence,

$$s = (1-\tau)^{1/2}$$

will give the singular point.

In the second branch, we can find the following singular point by the same way: When $\tau > 1/2$, we have $s = (1-\tau)^{1/2}$ as the singular point, and when $\tau < 1/2$, $s = \tau^{1/2}$.

These are results obtained from detailed consideration made by Montroll. Now that the expression of the distribution function of each branch consists of two or more analytically different parts, it is absurd to try to represent them by only one analytical expression. If we pay attention to thereabove circumstances, the entire distribution function in each branch is easily calculated according to Houston's method.

Let us consider the second branch in the same case $\tau = 1/3$ as before. In the domain $0 \leq s \leq 1/\sqrt{3}$, Eq. (15) gives the correct expression. To obtain the expression beyond $s = 1/\sqrt{3}$, translating the origin of the coordinate system to the point (π, π), we rewrite Eq. (1) in the form

$$\begin{vmatrix} 2(1+\cos\varphi_1)+(1-\cos\varphi_1\cos\varphi_2)-6s^2 & \sin\varphi_1\sin\varphi_2 \\ \sin\varphi_1\sin\varphi_2 & 2(1+\cos\varphi_2)+(1-\cos\varphi_1\cos\varphi_2)-6s^2 \end{vmatrix} = 0.$$

By the same way as in **III**, from this secular equation, we have

$$\nu_L f_2(s) = (4\pi)^{-1}\left[6s\frac{\cos^{-1}(3s^2-1)}{[(3s^2-1)(2-3s^2)]^{1/2}} + 4\sqrt{6}\,\frac{\cos^{-1}\sqrt{3/2}\,s}{(1-3s^2/2)^{1/2}}\right], \tag{19}$$

where $1/\sqrt{3} \leq s \leq \sqrt{2/3}$, $\sqrt{2/3}$ being the largest value of s in the second branch. In the expression (19), as s tends to $\sqrt{2/3}$, $\nu_L f_2(s)$ approaches to $6(2/3)^{1/2}\pi^{-1}$, and it is in accordance with ($33a$) in M2.

We can obtain the expressions of the distribution function in the first branch by the same way:

$$\nu_L f_1(s) = \pi^{-1}\left[\frac{\sin^{-1}s}{(1-s^2)^{1/2}} + \frac{3s}{4}\,\frac{\sin^{-1}\left\{\frac{\sqrt{3}}{2}(1-Q)^{1/2}\right\}}{(Q+1/3)^{1/2}(1-Q)^{1/2}Q}\right];\ 0 \leq s \leq \sqrt{2/3}, \tag{13}$$

$$\nu_L f_1(s)=\pi^{-1}\Big[\frac{\sin^{-1}s}{(1-s^2)^{1/2}}+\frac{3s}{4}\frac{\sin^{-1}\Big\{\frac{\sqrt{3}}{2}(1-Q)^{1/2}\Big\}}{(Q+1/3)^{1/2}(1-Q)^{1/2}Q}+3s\frac{\sin^{-1}(R+1/4)^{1/2}}{R(R+1/4)^{1/2}(3/4-R)^{1/2}}$$

$$+\frac{3s}{4}\frac{\cos^{-1}(5-6s^2)}{[3(1-s^2)(3s^2-2)]^{1/2}}\Big];\ R=(9/4-3s^2)^{1/2},\ \sqrt{2/3}\leq s\leq\sqrt{3/2}, \qquad (20)$$

$$\nu_L f_1(s)=\pi^{-1}\Big[2\frac{\cos^{-1}s}{(1-s^2)^{1/2}}+\sqrt{3}\,s\frac{\cos^{-1}(6s^2-5)}{[(3s^2-2)(1-s^2)]^{1/2}}\Big];\ \sqrt{3/2}\leq s\leq 1, \qquad (21)$$

where, two additional terms in the right hand side of Eq. (20) are due to contributions of frequencies whose curves of equal frequency converge to $s=\sqrt{2/3}$, i.e., the point (π, π), and the terms of Eq. (21) are due to the ones whose curves of equal frequency converge to $s=1$, i.e., the points $(\pi, 0)$ and $(0, \pi)$. The approximation of Eq. (21) is not good because of the less symmetrical properties of the curves of equal frequency (cf. Fig. 5 in M2). As an example, at $s=1$ the expression (21) takes the value $8/\pi$, whereas it takes $4\sqrt{3}/\pi$ according to the exact solution; and in $s=\sqrt{3}/2$, $\nu_L f_1(s)$ tends exactly to infinity, but the value of Eq. (21) remains finite.

The distribution functions thus obtained are plotted in Figs. 5 and 6, being in good agreement with M2's exact solution.[9]

§ **VI.** As is well known, in a certain interval at low temperatures the specific heats due to lattice vibrations express an anomaly from that expected by the Debye model. Based upon the Born-von Kárman model, Blackman[6] found the existence of two peaks in the distribution function for the simple cubic lattice instead of one peak predicted by Debye model. What is essential with regard to the specific heats at low temperatures consists in more heaping of eigen-frequencies in low frequency region of the distribution function based on the Born-von Kárman's atomic model than the Debye's continuum one.

The specific heat, C_v, due to lattice vibrations can be expressed in the form

$$C_v=k\int_0^{\nu_L}\frac{(h\nu/kT)^2 f(\nu)}{(\exp(h\nu/kT)-1)^2}\exp(h\nu/kT)\,d\nu$$
$$=k\int_0^{\nu_L}\Big(\frac{h\nu}{kT}\Big)^2 f(\nu)[\exp(-h\nu/kT)+2\exp(-2h\nu/kT)+\cdots]d\nu. \qquad (22)$$

At very low temperatures we have only to retain the first term of the expression in parenthesis, so we have, introducing $h\nu/kT=\xi$,

$$C_v\propto\int_0^{\Theta/T}\varphi(\xi)\exp(-\xi)\,d\xi, \qquad (23)$$

where $\varphi(\xi)=\xi^2 f(\xi)$,

and $\Theta=h\nu_L/k$.

Therefore, the place of dominant contribution to C_v is given by the point corresponding to the maximum value of the integrand of Eq. (23). From this

evaluation, we have, as a rough estimation of the place of the frequency region responsible for the specific heats at temperature T,

$$\varphi'(\xi)/\varphi(\xi)=1.$$

If we retain only one term ν^2 in $f(\nu)$ (three dimensional case), we have

$$4\left(\frac{T}{\theta}\right)=\frac{\nu}{\nu_L}$$

as a very rough estimation.

Accordingly, we can understand that only low frequency region contributes to the specific heats at very low temperatures. That is, not *mean*, but *local* approximation of the distribution function plays an important part in the low temperature specific heats.

As the temperature rises, the specific heats do not become sensitive to the *shape* of the distribution function, but depend on the *moments* of the distribution. The series expansion of thermodynamical quantities due to lattice vibrations in the moments of the frequency distribution offers a good approximation. But, extending this to the low temperature region, we have erroneous results. This situation is invariable, when we calculate the thermodynamical quantities by making use of the distribution function expanded in termes of some moments. For the distribution function expressed in terms of some moments does not approximate low frequency parts from the following reason: the contributions of low frequency parts to the moments are small because of small value and poor density of eigen-frequencies in this part, therefore large frequency regions suppress small ones.

To summarize, Houston's

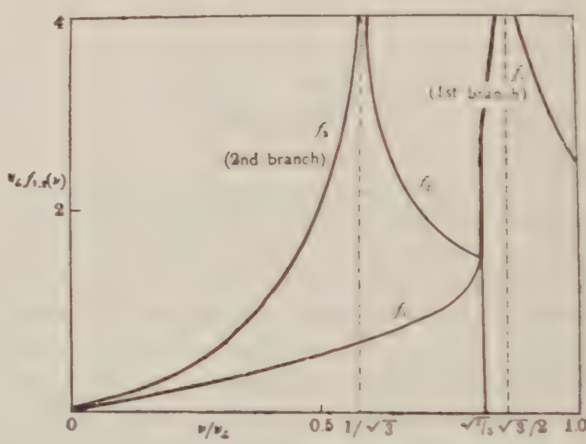

Fig. 5. The distribution function in each branch ($\tau=1/3$).

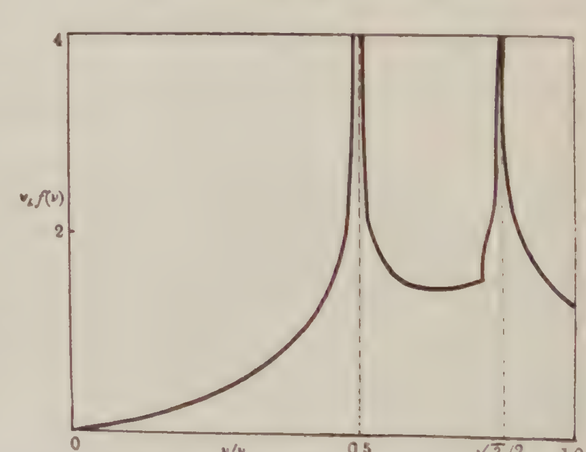

Fig. 6. The distribution function averaged over both branches ($\tau=1/3$).

method is useful for the evaluations of thermodynamical quantities due to lattice vibrations at low temperatures, while Thirring-Montroll's method is useful at high temperatures.

In conclusion, the writer wishes to express his cordial thanks to Assist. Prof. S. Ono for his critical reading of early drafts of the manuscript and to Mr. S. Huzinaga for his kind advice.

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9) Steeper slope in the neighborhood of the singular points of the distribution function obtained from Houston's method than the exact solution is due to the fact that the area circumscribed by curve of equal frequency is overestimated by the approximation in the neighborhood of the singular points as seen in Figs. 3 and 4. As a result, the approximated distribution function cannot be normalized by our procedure.

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Proton-Neutron Concentration Ratio in the Expanding Universe at the Stages preceding the Formation of the Elements.

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§ 1. Introduction.

In the theory of the origin of the elements by Gamow, Alpher, and colaborators[1], primordial matter (ylem) of the universe, which afterwards has been cooled down owing to the expansion of the universe and has formed the elements through nuclear reactions such as radiative capture and beta-decays, is assumed to consist solely of neutrons. At early stages, however, of high temperatures ($kT \gtrsim mc^2$, m being the electron mass) in the expanding universe before the formation of the elements, induced beta-processes caused by energetic electrons, positrons, neutrinos and antineutrinos, in addition to the natural decay of neutrons, such as

$$n+e^+ \rightleftarrows p+a\nu, \tag{1a}$$

$$n+\nu \rightleftarrows p+e^-, \tag{1b}$$

$$n \rightleftarrows p+e^-+a\nu, \tag{1c}$$

must have proceeded, their rates being faster at higher temperatures, and had a effect on the proton-neutron concentration ratio. At still higher temperatures $kT \gtrsim \mu c^2$ (μ is the mesons' mass), where large number of mesons are expected to be in existence, n-p conversion process induced by mesons would have been much more rapid owing to their stronger interactions with nucleons than the processes induced by light particles. Consequently, the n-p ratio must have been determined by the rates of such processes and those of changes in temperature and density in the universe resulting from its expansion.

We shall be based on the relativistic theory of the expanding universe, which are shown by Gamow as having a possibility to explain the origins both of the elements and the galactic nebulae.[1),5)] Then, the expansion and contraction rates of the universe at the stages of high compression are given by[2)]

$$\frac{1}{l}\frac{dl}{dt} = \pm\left(\frac{8\pi}{3}G\rho\right)^{\frac{1}{2}}, \tag{2}$$

where l is an arbitrary proper length of a volume containing a given amount of

matter and ρ the total mass density. We restrict ourselves in the case where radiational mass density is greater than material one,

$$\rho_r = aT^4/c^2 > \rho_m , \tag{3}$$

which is satisfied if $\rho_m < 1\,\mathrm{g/cm^3}$ at $T = 10^{9}\,{}^\circ\mathrm{K}$, including the case of Gamow and Alpher where the temperature and material density at the beginning of the elements formation are given by $T \simeq 10^{9}\,{}^\circ\mathrm{K}$ and $\rho_m = 10^{-6} \sim 10^{-8}\,\mathrm{g/cm^3}$. In this case, conservations of energy and matter give for the variations of temperature and material density, for $kT < Mc^2$ (M being the nucleon mass),

$$T \sim 1/l, \quad \text{and} \quad \rho_m \sim 1/l^3. \tag{4}$$

We shall study the changes in the concentrations of proton, neutron, electron, positron, photon, neutrino, and antineutrino, which are all assumed to be free, in conditions of varying temperature and density which are determined macroscopically by Eqs. (2) and (4). It must be noticed that, when we consider a volume containing a given mass of matter in the universe which is isotropic and homogeneous, numbers of neutrinos and antineutrinos which are missed through the boundaries are just gained from the surroundings, and then we must take into account their effects on the beta-processes and on the statistics of all the particles in the case of equilibrium.

In § 2, rates of reactions between those particles at constant temperature and density are studied and it is shown that, except for the beta-processes, they are very fast compared with the rates of the temperature change of the universe. Accordingly, these particles, for instance electron pairs, can be treated as in thermodynamical equilibrium with radiations. In § 3, rates of beta-processes are calculated, and in § 4 we obtain equations which determine the concentrations of neutron, proton, neutrino, and antineutrino, taking into account the contraction or expansion of the universe. These equations are reduced to a simple form due to a circumstance that concentrations of neutrinos are much larger than that of nucleons in the condition (3), in § 5. At temperatures higher than about $2 \times 10^{10}\,{}^\circ\mathrm{K}$ thermodynamical equilibrium is being continually established, but at lower temperatures induced beta-processes are largely reduced owing to lack of energetic light particles and there occurs a time lag in the n-p ratio departing from its equilibrium value, or, in other words, n-p ratio becomes "frozen" when the expansion of the universe becomes faster than the beta-processes. The numerical results show that, for the universe satisfying the condition (3), n-p ratio at the beginning of the elements formation is nearly 1 : 4 almost irrespective of its initial values as long as initial temperatures are so high ($T \gtrsim 2 \times 10^{10}\,{}^\circ\mathrm{K}$) that equilibrium has once been attained. Further remarks are discussed in § 6.

§ 2. Rates of Processes at High Temperatures.

Reactions between nucleons, electrons, photons, and neutrinos proceed in the

direction to approach the statistical equilibrium. For the sake of simplicity these particles are all assumed to be free. We compare rates of processes (Coulomb-, Compton-, and nuclear scatterings, pair creations, etc.) at constant temperature and density with that of the temperature change in the universe which is given by Eq. (2). The latter is conveniently characterized by the time during which the temperature changes by its order of magnitude. Putting in Eq. (2) $\rho=aT^4/c^2$, we have

$$\tau_T=T\Big/\left|\frac{dT}{dt}\right|=\left(\frac{8\pi}{3}\frac{Ga}{c^2}\right)^{-\frac{1}{2}}T^{-2}. \tag{5}$$

In the following we consider the case where the temperature is not so high as to allow of the existence of mesons but is higher than that at the time of the elements formation, i. e., $10^{12}>T>5\times10^{9}\,°\mathrm{K}$. An average life of any one particle for a single collision or reaction with other particles having concentration n_i is given by $1/v\sigma n_i$, where v is the relative velocity corresponding to the temperature and σ is the cross section. Such lives for nuclear scattering between nucleons, for Coulomb scattering between charged particles, and for electromagnetic processes such as Compton scattering and pair creation between photons and electron pairs are found to be much shorter than τ_T in the above temperature range and in the universe in which condition (3) is satisfied as long as ρ_m is greater than $10^{-10}\,\mathrm{g\,cm^3}$ at $T=10^{9}\,°\mathrm{K}$. Consequently, we can consider that these particles and photons obey the Maxwell's and Planck's distribution laws at temperature T, respectively, and are in thermodynamical equilibrium with each other, apart from the concentrations of neutron and proton and the energy distributions and concentrations of neutrino and antineutrino, which are controlled by beta-processes having much smaller cross sections.

As an important example the rate of reactions between photons and electron pairs is shown below. At such high temperatures electron pairs are found to be created exclusively by two quanta collisions

$$h\nu+h\nu' \rightarrow e^{+}+e^{-}.$$

The number of pairs created per cm³ per sec is given by

$$\frac{dn_{pair}}{dt}=\frac{1}{2c}\iint d\Omega d\Omega'\iint d\nu d\nu'\frac{B(\nu)}{h\nu}\frac{B(\nu')}{h\nu}\sin^2\frac{\psi}{2}\,\sigma\left(h\nu^{\frac{1}{2}}\nu'^{\frac{1}{2}}\sin\frac{\psi}{2}\right),$$

$$B(\nu)=2h\nu^3c^{-2}(e^{h\nu/kT}-1)^{-1},\quad \sigma(h\nu)=2\pi\left(\frac{e^2}{mc^2}\right)^2(2\theta C^{-2}+2\theta C^{-4}$$

$$-\theta C^{-6}-SC^{-3}-SC^{-5}),\quad C=\frac{h\nu}{mc^2}=\cosh\theta,\quad S=\sinh\theta.$$

where Ω is a solid angle, ψ is the angle between the directions of propagation of the two quanta, and σ is the cross section obtained by Breit and Wheeler.[3] After elementary calculations we have

$$\frac{dn_{pair}}{dt} \simeq \pi c \left(\frac{e^2}{mc^2}\right)^2 \left(\frac{kT}{hc}\right)^6 \times \begin{cases} 8\pi^3 \left(\frac{mc^2}{kT}\right)^3 e^{-2mc^2/kT}, & (kT \ll mc^2), \\ \frac{8}{9}\pi^6 \left(\frac{mc^2}{kT}\right)^2 \left(\log \frac{2kT}{mc^2} - 1\right), & (kT \gg mc^2). \end{cases} \tag{6}$$

The order of magnitude of the time required to attain equilibrium is given by $n_{pair,eq} / \frac{dn_{pair}}{dt}$, where $n_{pair,eq}$ is the number of pairs in equilibrium which is given by Eqs. (7), (8) and (9) below, and this is found to be much smaller than τ_p of the universe. Thus, electron pairs are continually in equilibrium with radiations, following the change in temperature without any time lag.

Numbers of electrons and positrons per cm^3 are given by the theory of statistics as

$$n_{e^\mp} = \frac{8\pi}{h^3}\int_0^\infty e^{\pm\lambda - E/kT} p^2 dp = 8\pi\left(\frac{kT}{hc}\right)^3 x^2 K_2(x) e^{\pm\lambda}, \tag{7}$$

where

$$K_n(x) = \frac{\pi i}{2} e^{\frac{1}{2}n\pi i} H_n^{(1)}(ix) = \begin{cases} \left(\frac{\pi}{2x}\right)^{\frac{1}{2}} e^{-x}\left\{1 + \frac{4n^2-1}{8x} + \cdots\right\}, & (x \gg 1), \\ \frac{1}{2}\left\{\frac{(n-1)!}{(x/2)^n} - \frac{(n-2)!}{(x/2)^{n-2}} + \cdots\right\}, & (x \ll 1), \end{cases}$$

$$x = mc^2/kT, \tag{8}$$

and λ is determined by the condition that the total charges are zero, i.e., $n_{e^-} - n_{e^+} = n_p$, n_p being the number of protons per cm^3,

$$\lambda = \sinh^{-1}\left\{n_p / 16\pi\left(\frac{kT}{hc}\right)^3 x^2 K_2(x)\right\}. \tag{9}$$

In our case λ is found to be very small compared with unity, confirming that degeneracies of electrons and positrons can be neglected.

§ 3. Rates of Beta-processes.

According to the Fermi's theory of beta-decay, processes, which change neutrons to protons and *vice versa* and are energetically possible, are given by (1a), (1b), and (1c). In the following, energy distributions of neutrinos and antineutrinos are assumed to obey the Maxwell's law, apart from their concentrations, for the following reasons: at temperatures higher than 10^{12}°K they will be in thermodynamical equilibrium through rapid processes involving the mesons; in view of their motions in the expanding universe given by the theory of relativity,[2)] they obey the Maxwell's law continually if they do initially and their numbers do not change in course of time; and in our case numbers of neutrinos and antineutrinos can be considered as much larger than that of nucleons, as shown later. Then, their energy distributions are expressed as

$$n_{a\nu}^{\nu}(E)dE=\frac{n_{a\nu}^{\nu}}{2(kT)^3}e^{-E/kT}E^2dE\,, \tag{10}$$

where n_ν and $n_{a\nu}$ are their numbers per cm^3 and their rest mass is assumed to be zero.

Numbers of reactions per cm^3 per sec in the leftward and rightward processes in (1a) ; (1b) ; and (1c) can be written as $An_pn_{a\nu}$, $A'n_nn_{e^+}$; $Bn_pn_{e^-}$, $B'n_nn_\nu$; and $Cn_pn_{e^-}n_{a\nu}$, $C'n_n$, respectively. In particular C' is the decay constant of the neutron. $An_pn_{a\nu}$ can be calculated in the following way.[4] The probability that an antineutrino of energy E is captured by protons per sec. is given by

$$w_{a\nu}(E)=(g^2n_p/2\pi\hbar^4c^3)cp_+E_+,\quad E_+=(m^2c^4+c^2p_+^2)^{\frac{1}{2}}=E-Q, \tag{11a}$$

$$Q=M_nc^2-M_pc^2\,, \tag{12}$$

where g is the Fermi's interaction constant and E_+ is the energy of the emitted positron. Then, we have

$$An_pn_{a\nu}=\int_{Q+mc^2}^{\infty}w_{a\nu}(E)n_{a\nu}(E)dE\,, \tag{13a}$$

where $n_{a\nu}(E)$ is given by Eq. (10). For $A'n_nn_{e^+}$ we have in the same way,

$$w_{e^+}(E)=(g^2n_n/2\pi\hbar^4c^3)E_{a\nu}^2,\quad E_{a\nu}=E+Q\,, \tag{11a'}$$

$$A'n_nn_{e^+}=\int_{mc^2}^{\infty}w_{e^+}(E)\,n_{e^+}(E)dE, \tag{13a'}$$

where $n_{e^+}(E)$ is given by Eq. (7). In the same way we obtain B and B',

$$w_{e^-}(E)=(g^2n_p/2\pi\hbar^4c^3)E_\nu^2,\quad E_\nu=E-Q\,, \tag{11b}$$

$$Bn_pn_{e^-}=\int_{Q}^{\infty}w_{e^-}(E)\,n_{e^-}(E)dE\,; \tag{13b}$$

$$w_\nu(E)=(g^2n_n/2\pi\hbar^4c^3)cp_-E_-,\quad E_-=E+Q\,, \tag{11b'}$$

$$B'n_nn_\nu=\int_0^{\infty}w_\nu(E)n_\nu(E)dE\,. \tag{13b'}$$

To calculate C and C' we consider antineutrinos as holes in the negative energy states of neutrino,

$$n+\nu^*\rightleftarrows p+e^-,$$

where asterisk denotes negative energy states. The probability that an electron of energy E is captured by protons per sec is given by

$$w_{e^-}(E)=(g^2n_p/2\pi\hbar^4c^3)\,E_{\nu*}^2\times\frac{n_{a\nu}}{2(kT)^3}e^{-E_{a\nu}/kT}E_{a\nu}^2\Big/\frac{8\pi}{h^3c^3}E_{a\nu}^2\,, \tag{11c}$$

$$E_{\nu*}=-E_{a\nu}=E-Q<0\,,$$

where the last factor in $w_{e^-}(E)$ means the probability that the state with negative

energy $E_{\nu *}$ is vacant, i. e., anti-neutrino of positive energy $E_{a\nu}$ is present. Then, we have

$$Cn_p n_{e^-} n_{a\nu} = \int_{mc^2}^{Q} w_{e^-}(E) n_{e^-}(E) dE \,. \tag{13c}$$

C' for the reverse process can be obtained from C by the principle of detailed balancing, but we have directly

$$w_{\nu *}(E^*) = (g^2 n_n / 2\pi \hbar^4 c^3) c p_- E_-,$$
$$E_- = Q + E^* = Q - E_{a\nu}, \tag{11c'}$$

$$C' n_n = \int_{mc^2-Q}^{0} w_{\nu *}(E^*)\,(8\pi/h^3c^3) \times E^{*2} dE^*, \tag{13c'}$$

where $(8\pi/h^3c^3)E^{*2}$ is the density of the negative energy states E^*.

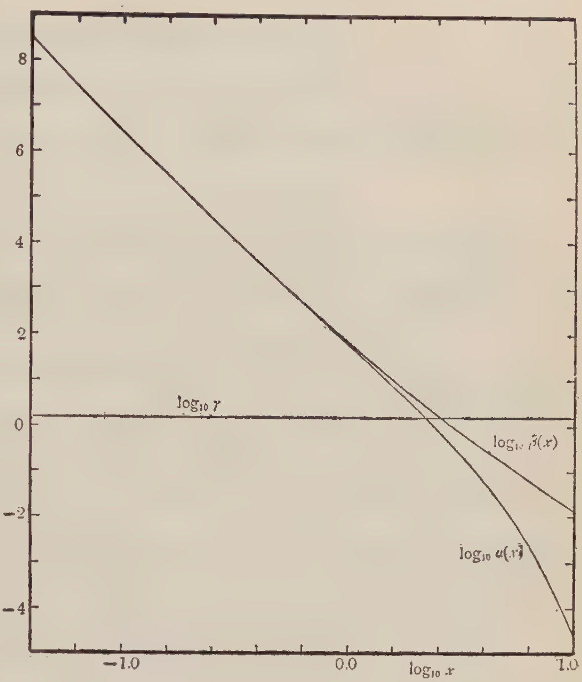

Fig. 1.

Accordingly, we obtain A, A', etc. as functions of temperature only,

$$A = \frac{1}{\tau_0}\frac{1}{8\pi}\left(\frac{hc}{kT}\right)^3 \frac{1}{2} e^{-qx}\alpha(x), \qquad A' = \frac{1}{\tau_0}\frac{1}{8\pi}\left(\frac{hc}{kT}\right)^3 \frac{1}{x^2 K_2(x)}\alpha(x)\,; \tag{14a}$$

$$B = \frac{1}{\tau_0}\frac{1}{8\pi}\left(\frac{hc}{kT}\right)^3 \frac{1}{x^2 K_2(x)} e^{-qx}\beta(x), \quad B' = \frac{1}{\tau_0}\frac{1}{8\pi}\left(\frac{hc}{kT}\right)^3 \frac{1}{2}\beta(x)\,; \tag{14b}$$

$$C = \frac{1}{\tau_0}\frac{1}{8\pi}\left(\frac{hc}{kT}\right)^3 \frac{1}{x^2 K_2(x)} \frac{1}{8\pi}\left(\frac{hc}{kT}\right)^3 \frac{1}{2} e^{-qx}\gamma, \quad C' = \frac{1}{\tau_0}\gamma\,; \tag{14c}$$

where

$$q = Q/mc^2, \ \tau_0 = 2\pi^3\hbar^7/g^2 m^5 c^4, \tag{15}$$

$$\alpha(x) = \int_1^\infty e^{-xy}(y+q)^2(y^2-1)^{\frac{1}{2}} y\,dy = \frac{3K_3(x)}{x^2} + \frac{K_2(x)}{x} + 2q\left\{\frac{3K_2'(x)}{x^2} + \frac{K_1(x)}{x}\right\} + q^2\frac{K_2'(x)}{x}$$

$$= \begin{cases} 24x^{-5}(1 + \frac{1}{2}qx + \frac{1}{12}q^2x^2) - x^{-3}(1 + qx + \frac{1}{2}q^2x^2) + \cdots, & (x \ll 1) \\ (\pi/2)^{\frac{1}{2}}(q+1)^2 x^{-\frac{3}{2}} e^{-x} + \cdots, & (x \gg 1)\,; \end{cases} \tag{16a}$$

$$\beta(x) = \int_0^\infty e^{-xy}\{(y+q)^2 - 1\}^{\frac{1}{2}}(y+q)y^2 dy = \begin{cases} 24x^{-5}(1 + \frac{1}{2}qx + \frac{1}{12}q^2x^2) - x^{-3} + \cdots, \\ \qquad (x \ll 1) \\ 2q(q^2-1)^{\frac{1}{2}}x^{-3} + \cdots, \quad (x \gg 1) \cdot \end{cases} \tag{16b}$$

$$\gamma=\int_1^q (q-y)^2(y^2-1)^{\frac{1}{2}}\,y\,dy. \tag{16c}$$

When we put $q=2.5$, γ becomes 1.51. Values of $\alpha(x)$ and $\beta(x)$ which are calculated for $q=2.5$ are shown in Fig. 1.

§ 4. Reactions in the Expanding Universe.

Changes in the concentrations of neutron, proton, neutrino, and antineutrino in the contracting or expanding universe are given by the following equations. Here, we consider the two cases:

(i) neutrinos and antineutrinos are physically distinguishable, say by their magnetic moments,

(ii) they are identical, say the form of the Hamiltonian for the beta-decay is symmetrical in neutrino and antineutrino.

In the case (i) these equations are

$$\frac{d}{dt}\left(\frac{n_p}{\rho_m}\right)=\frac{1}{\rho_m}\left(-An_pn_{a\nu}+A'n_nn_{e^+}-Bn_pn_{e^-}+B'n_nn_\nu-Cn_pn_{e^-}n_{a\nu}+C'n_n\right), \tag{17.1}$$

$$\frac{d}{dt}\left(\frac{n_\nu}{\rho_m}\right)=\frac{1}{\rho_m}\left(Bn_pn_{e^-}-B'n_nn_\nu\right),$$

with two conditions

$$(n_n+n_p)/\rho_m=1/M, \text{ and } (n_p+n_\nu-n_{a\nu})/\rho_m=\text{constant}, \tag{18.1}$$

where the relation between t and T is given by Eqs. (2) and (4).
In the case (ii) they are

$$\frac{d}{dt}\left(\frac{1}{\rho_m}\begin{Bmatrix}n_p\\ n_\nu\end{Bmatrix}\right)=\frac{1}{\rho_m}\left(-An_pn_\nu+A'n_nn_{e^+}\mp Bn_pn_{e^-}\pm B'n_nn_\nu-Cn_pn_{e^-}n_{a\nu}+C'n_n\right), \tag{17.2}$$

where upper and lower signs correspond to equations for n_p and n_ν, respectively, with a condition that

$$(n_n+n_p)/\rho_m=1/M. \tag{18.2}$$

Introducing non-dimensional variables

$$y=\frac{n_p-n_n}{n_p+n_n}, \quad \text{and} \quad z_{a\nu}^{\nu}=n_{a\nu}^{\nu}/16\pi\left(\frac{kT}{hc}\right)^3, \tag{19}$$

and using Eq. (7), above equations become, in the case (i)

$$\tau_0\frac{dy}{dt}=\alpha(x)\{(1-y)e^{-\lambda}-(1+y)z_{a\nu}e^{-qx}\}+\beta(x)\{(1-y)z_\nu-(1+y)e^{\lambda-qx}\}$$
$$+\gamma\{(1-y)-(1+y)z_{a\nu}e^{\lambda-qx}\}, \tag{20.1}$$

$$R\tau_0\frac{dz_\nu}{dt}=-\beta(x)\{(1-y)z_\nu-(1+y)e^{\lambda-qx}\},$$

$$z_{a\nu}-z_\nu=(y+1)/R+z_c, \tag{21.1}$$

where z_c is a constant which is determined by initial conditions, and in the case (ii)

$$\tau_0\frac{dy}{dt}=\alpha(x)\{(1-y)e^{-\lambda}-(1+y)ze^{-qx}\}+\beta(x)\{(1-y)z-(1+y)e^{\lambda-qx}\}$$
$$+\gamma\{(1-y)-(1+y)ze^{\lambda-qx}\}, \tag{20.2}$$

$$R\tau_0\frac{dz}{dt}=\alpha(x)\{\cdots\quad\}-\beta(x)\{\cdots\quad\}+\gamma\{\cdots\quad\},$$

with

$$R=\frac{2M}{\rho_m}16\pi\left(\frac{kT}{hc}\right)^3=\frac{15}{4\pi^4}\frac{\rho_r}{\rho_m}\frac{Mc^2}{kT}, \tag{22}$$

which measures the ratio of concentrations of photon (or, as shown later, neutrino and antineutrino) and nucleon, and is a very large constant for the universe satisfying the condition (3). It is shown in the next section that these somewhat complicated equations can be simplified substantially to a linear form.

§ 5. Solutions for the N-P Ratio.

First the equilibrium states are considered. If the change of temperature in the universe is much slower than the beta-processes considered above, energy distributions and concentrations of all the particles vary in accordance with their equilibrium values corresponding to a temperature and density at each stage. In this case detailed balancings are always established in all the processes, and in particular curly brackets on the righthand sides of Eqs. (20.1) or (20.2) must all vanish simultaneously. Then, we have in the case (i)

$$n_n/n_p\equiv(1-y)/(1+y)=e^{\lambda-Q/kT}/z_\nu,$$

$$z_\nu\cdot z_{a\nu}=1,\quad\text{and}\quad z_{a\nu}-z_\nu=(1+y)/R+z_c. \tag{23.1}$$

In view of the facts that in our case $0\leq(1+y)/R\leq 2/R\ll 1$ and that, in the hole theory of the neutrino, $n_{a\nu}$ is the number of holes in the negative energy states, it will be natural to assume that $z_c\approx 0$ and to take $z_{a\nu}\approx z_\nu$ since $z_{a\nu}$ and z_ν can not be both smaller than unity. In such a case Eqs. (23.1) reduce to the following ones which are obtained in the case (ii):

$$n_n/n_p\equiv(1-y)/(1+y)=e^{\lambda-Q/kT},\quad\text{and}\quad z=1. \tag{23.2}$$

In solving Eqs. (20.1) or (20.2) which are general as yet, we put $\lambda=0$ since λ is smaller than 2×10^{-4} at $T=10^9$°K even if ρ_m is as large as 1 g/cm³ and much smaller at higher temperatures, as shown by Eq. (9), and further we

take $z_\nu = z_{a\nu} (=z)$ in the case (i) since they do not change appreciably after the equilibrium state has once been attained, as shown later. From these equations and Fig. 1 we see that times during which y and z change by their orders of magnitude at given high temperatures $(x<1)$ are given by

$$\tau_y = \tau_0/a(x), \quad \text{and} \quad \tau_z = R\tau_0/a(x), \tag{24}$$

respectively. At temperatures where they are both smaller than τ_T given by Eq. (5), equilibrium states, which are considered above, will be attained immediately even if there are initially appreciable deviations from them. Such temperatures are given by

$$1/x \equiv kT/mc^2 \geq 2 \times R^{\frac{1}{5}}. \tag{25}$$

Further, at temperatures higher than 10^{12}°K, z will attain its equilibrium value much faster through processes involving mesons which are in existence in much larger numbers (almost the same numbers as of photons) than nucleons.

It is shown, as follows, that we can put $z=1$ (equilibrium value) throughout in solving Eqs. (20.1) or (20.2). Expanding y and z in powers of $1/R$,

$$y = y_0 + y_1/R + y_2/R^2 + \cdots, \quad \text{and} \quad z = z_0 + z_1/R + z_2/R^2 + \cdots, \tag{26}$$

and equating the terms with same powers of $1/R$ in Eqs. (20.1) or (20.2), we obtain first

$$dz_0/dt = 0. \tag{27}$$

Putting $z_0 = 1$ further, we have

$$\tau_0 dy_0/dt = -\{y_0 - \tanh(qx/2)\}(1+e^{-qx})(a+\beta+\gamma), \tag{28}$$

$$\tau_0 dz_1/dt = \{y_0 - \tanh(qx/2)\}(1+e^{-qx}) \times \begin{cases} \beta, & \text{in the case (i)}, \quad (29.1) \\ (\beta - a - \gamma), & \text{in the case (ii)}, \quad (29.2) \end{cases}$$

$$\tau_0 dy_1/dt = -y_1(a+\beta+\gamma)(1+e^{-qx}) + z_1\{\beta(1-y_0) - (a+\gamma)(1+y_0)e^{-qx}\}, \text{ etc.} \tag{30}$$

From these equations one can see that above expansions are good approximations in our case where R is a very large constant. This corresponds to a circumstance that numbers of neutrinos and antineutrinos, which are equal to that of photons in orders of magnitude in the case $z=1$, are far greater than that of nucleons in the universe satisfying the condition (3). Accordingly, the n-p ratio is almost completely determined by Eq. (28) and there are no essential differences between the cases (i) and (ii).

In order to obtain an accurate relation between t and x, we must include in the total mass density those contributions from neutrinos and electron pairs in addition to that from photons. They are given by, from Eqs. (10) and (7), as

$$\rho^\nu_{a\nu} = \frac{1}{c^2} \int \frac{n^\nu_{a\nu}}{2(kT)^3} e^{-E/kT} E^3 dE = \frac{48\pi}{c^2} \frac{(kT)^4}{(hc)^3} z^\nu_{a\nu}, \tag{31}$$

$$\rho_e^{\mp}=\frac{8\pi}{(hc)^3}\int e^{\pm\lambda-E/kT}Ep^2dp=\frac{48\pi^{\bullet}}{c^2}\frac{(kT)^4}{(hc)^3}e^{\pm\lambda}J(x), \tag{32}$$

where

$$J(x)=x^3\left(\frac{K_3(x)}{8}+\frac{K_1(x)}{24}\right)=\begin{cases}1-\frac{1}{12}x^2+\cdots & ,\ (x\ll 1),\\ \left(\frac{\pi}{2}\right)^{\frac{1}{2}}x^{\frac{5}{2}}e^{-x}\left(\frac{1}{6}+\frac{9}{16}\frac{1}{x}+\cdots\right), & (x\gg 1).\end{cases} \tag{33}$$

Since n-p ratio is mainly determined in the region $x\ll 1$, we put in further calculations $J(x)=1$, and $\lambda=0$ as before. Then, the total mass density is expressed as

$$\rho=(1+r)aT^4/c^2, \tag{34}$$

$$r=\begin{cases}4\times 90/\pi^4 & \text{in the case (i)}, \quad (35.1)\\ 3\times 90/\pi^4 & \text{in the case (ii)}. \quad (35.2)\end{cases}$$

From Eqs. (2) and (4) we obtain

$$x^2=\frac{|t|}{t_m},\ t_m=\frac{1}{2}\left(\frac{k}{mc^2}\right)^2\left\{\frac{8\pi}{3}\frac{Ga}{c^2}(1+r)\right\}^{-\frac{1}{2}}=6.6(1+\gamma)^{-\frac{1}{2}}\text{sec.} \tag{36}$$

Then, Eq. (28) becomes, omitting a suffix in y,

$$\frac{dy}{dx}=\pm\frac{2t_m}{\tau_0}x(1+e^{-qx})(\alpha(x)+\beta(x)+\gamma)(\tanh\frac{qx}{2}-y),\ \text{for}\ t\lesseqgtr 0. \tag{37}$$

Using a function with an arbitrary constant x_0,

$$f(x)=(2t_m/\tau_0)\int_{x0}^{x}(1+e^{-qx})(\alpha(x)+\beta(x)+\gamma)x\,dx, \tag{38}$$

general solutions of Eq. (37) are written as follows:

in the case $t<0$, i. e., the universe being contracting,

$$y(x)=y_-(x)+Ae^{f(x)},$$

$$y_-(x)=e^{f(x)}\int_x^{\infty}e^{-f(x)}\frac{df(x)}{dx}\tanh\frac{qx}{2}dx, \tag{39}$$

in the case $t>0$, i. e., the universe being expanding,

$$y(x)=y_+(x)+Be^{-f(x)},$$

$$y_-(x)=e^{-f(x)}\int_0^{x}e^{f(x)}\frac{df(x)}{dx}\tanh\frac{qx}{2}dx, \tag{40}$$

where A and B are arbitrary constants, and y_- and y_+ are the particular solutions which are unity at infinity, tend to $\tanh(qx/2)$ as x approaches to zero, and are finite everywhere.

Numerical calculations are carried out by taking $q=2.5$ and $\tau_0/\gamma=30$ min.,

and using the value of r given by Eq. (35.2), there being no essential difference between using Eqs. (35.1) and (3.52) in view of the uncertainty in the value of τ_0/γ (decay life of neutron). The results are shown in Fig. 2 together with the equilibrium value $y_{eq}=\tanh(qx/2)$. In the region $x \lesssim 0.1$, general solutions tend to y_- and y_+ rapidly, with a consequence that n-p ratio at lower temperatures in the case $t>0$ are almost entirely independent of its initial values. With decreasing temperature the rate of induced beta-processes, i. e., $\alpha(x)+\beta(x)$, is largely reduced owing to decreases in numbers of energetic electrons and neutrinos, and there occur departures in the n-p ratio from its equilibrium value ("freezing" sets in). The value of x, where these deviations become appreciable, is roughly estimated by putting $\tau_y=\tau_T$, where τ_y and τ_T are given by Eqs. (24) and (5), and is found to be nearly 0.5. In the region $x>3$, γ becomes larger than $\alpha(x)+\beta(x)$ and natural decay of neutrons is the main process.

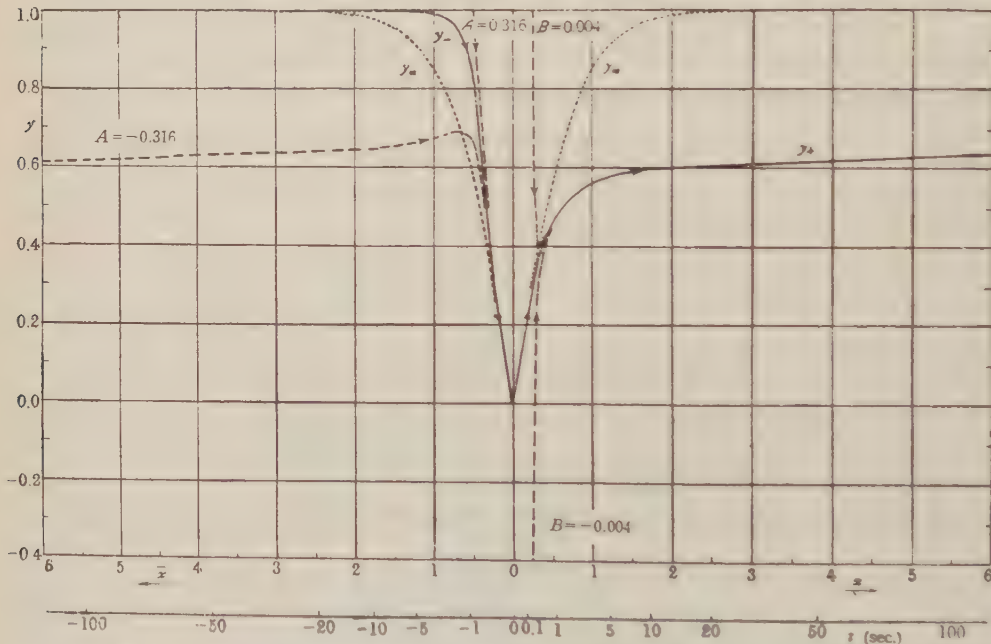

Fig. 2. n-p ratios as functions of $x=m_ec^2/kT$ and time t of the universe. Dotted line is equilibrium value: $y_{eq}=\tanh(qx/2)$, and broken curves show $y_- + Ae^{-f(x)}$ and $y_+ + Be^{-f(x)}$ where x_0 is taken as unity in Eq. (38).

§ 6. Concluding Remarks.

In the universe in which $\rho_r > \rho_m$ and at high temperatures where elements are not allowed to be in existence, the n-p ratio follows the curves shown in Fig. 2. It can be seen in particular that, if the existing laws of physics, microscopic and macroscopic, are valid at least up to temperatures $\sim 2\times 10^{10}\,{}^\circ$K, the n-p

ratio at the beginning of the elements formation, i. e., at $x \gtrsim 1$, is nearly $1:4$, whatever the physical conditions at higher temperatures, especially at the epoch $t=0$ when the universe is singular according to the current theory, may be.

It is known that at present hydrogen and helium together form about 97 percent of all matter. If we assume that formation of nuclei heavier than He^4 can be neglected, and that reactions involving beta-processes such as $n \to p+e^-$, $p+p \to H^2 + e^+$, and $H^3 \to He^3 + e^-$, which are much slower than other nuclear transmutations such as gamma-ray or particle emission unless material density is extremely low, are not effective during the formation process, He^4 is built up from original neutrons and protons, after all, as $2n+2p \to He^4$, whatever the routes of formation may be, for instance $n+p \to H^2$, $H^2+H^2 \to H^3+p$, $H^3+H^2 \to He^4+n$, or $n+p \to H^2$, $H^2+n \to H^3$, $H^3+p \to He^4$. Consequently, the hydrogen-helium abundance ratio (in number) resulting from the initial n-p ratio $1:4$ becomes $6:1$, whereas recent observed values in stellar atmospheres and meteorites range from $5:1$ to $10:1$.

Under the original assumption of Gamow that "ylem" consists solely of neutrons, it is difficult to explain the fact that the building-up processes of the elements jump over the "crevasses" of unstable mass numbers 5 and 8, as shown by Fermi and Turkevitch.[5] However, the existence of an appreciable amount of original protons may relieve this situation owing to increasing probabilities of capture by light nuclei of proton, deuteron, triton, and helium nuclei, since the formation process is expected to begin at higher temperatures. Such a situation, together with the above conclusion on the hydrogen and helium abundances, seems to encourage further calculations on the formation of light elements, at least up to C^{12}. In this connection, to treat the building-up processes at initial stages, especially formation of deuterons, more accurately, it will be necessary to take into account the effects of their reverse processes, for instance $A_z + n \leftarrow (A+1)_z + h\nu$, caused by radiations having high concentrations at high temperatures.

In conclusion, the author wishes to express his cordial thanks to Prof. H. Yukawa for his continual interests and advices, to Prof. G. Gamow for his kind suggestions and advices, and to Prof. Z. Shirogane for his continual encouragements.

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Note on the Self-Energy and Self-Stress. II.

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In the first part of this paper, we treated the general property of self-energy and self-stress of Fermion and Boson in usual perturbation formalism, and found the general relation*

$$\langle \overset{\circ}{T}_{\mu\mu} \rangle = m\frac{\partial}{\partial m}\langle \overset{\circ}{T}_{44}\rangle + \mu\frac{\partial}{\partial \mu}\langle \overset{\circ}{T}_{44}\rangle$$

between them, which holds for Fermion with mass m interacting with Boson with mass μ, and vice versa. The fact that the self-stress can be expressed as surface-integral in k^4-space was shown in the last paragraph. We now want to show that they can be reduced to the two dimensional surface in $k^{(3)}$-space, and that the numerical value of them depends on the shape of surface taken. The same results as in I can be obtained by choosing the surface spherically. And it was also shown that the self-stress only contains as diverging quantity the multiple of the surface area of $k^{(3)}$-space, no logarithmic divergences. In § 3, a physical interpretation of expectation value of stress-tensor, which has analogous meaning as classical Poincare tensor, is given. And, finally, we want to seek for the relation between convergence condition of self-energy and stability condition of particle from the three different standpoints, founding that the formalistic regularization is to be modified to be effective for stabilization problem.

§ 1. Self-Stress of Fermion due to Boson field.

The self-stress of Fermion, i. e., spin 1/2 particle obeying Dirac equation, due to the Boson can be obtained as a transition matrix element of Feynman diagram through the stress tensor. (The Boson means only scalar, pseudoscalar, and vector neutral mesons with mass μ in weaker coupling with Fermion, here.)

Since this tensor is composed with three parts: $T_{\mu\nu}^{(\text{particle})} + T_{\mu\nu}^{(\text{field})} + T_{\mu\nu}^{(\text{interaction})}$, the transition can be classified into three parts: (Owing to the difference in notation, the results of this paper is slightly changed from perturbation calculation, in I, i. e., the self-stress is multiplied by -1 and momentum by i) :[1]

a) Transition through $T_{\mu\nu}^{(\text{particle})}$:

$T_{\mu\nu}^{(\text{particle})}$ has, in general, the following form: (in ordinary notation used in I)

* In these dissussions of Self Stress of Fermion, we should consider for the scalar-meson interaction that the polarization term is put out of consideration. (see § 1 (d). Page 241).

$$T_{\mu\nu}^{(\text{particle})} = \frac{1}{2}\left(\Psi^{\dagger}\gamma^{\mu}\frac{\partial \Psi}{\partial x_{\nu}} - \frac{\partial \Psi^{\dagger}}{\partial x_{\nu}}\gamma^{\mu}\Psi\right) \tag{1}$$

the transition through this can also be separated into three parts:

a, 1) The transition as indicated in Fig. 1. If one traces events in time, one can obtain the term in perturbation theory. But, translating this into Feynman notation, one should note that the differential operators contained in (1) do some another contribution due to the jump in the propagation factor $K_+(X)$ of Feynman, and one should subtract this part from the expression:

$$ig^2\int_{(1)}\int_{(3)} S_+(3,1)e^{ipx_3}\gamma^{\cdot\cdot}K_+(3,2)\,(-i)\frac{1}{2}\gamma^{\mu}(\overrightarrow{\nabla}-\overleftarrow{\nabla})_{\nu 2}K_+(2,1)\gamma^{\cdot\cdot}e^{-ipx_1}\,d\tau_1 d\tau_3$$
$$-g^2\int_{(3)} S_+(3,1)\,e^{ipx_3}\gamma^{\cdot\cdot}K_+(3,1)\gamma^{\cdot\cdot}e^{-ipx_1}\cdot d\tau_3\cdot\delta_{\mu 4}\delta_{\nu 4}. \tag{2}$$

The subtracted term is just $\langle \bar{T}_{44}\rangle$. Writing this in momentum space, we obtain:

$$-\frac{g^2}{\pi i}\int\gamma^{\cdot\cdot}(\bar{p}-\bar{k}-m)^{-1}(-1)\gamma^{\mu}(p_{\nu}-k_{\nu})\,(\bar{p}-\bar{k}-m)^{-1}\gamma^{\cdot\cdot}\frac{1}{k^2-\mu^2}d^{(4)}k$$
$$-\frac{g^2}{\pi i}\int\gamma^{\cdot\cdot}(\bar{p}-\bar{k}-m)^{-1}\gamma^{\cdot\cdot}\frac{1}{k^2-\mu^2}d^{(4)}k\cdot\delta_{\mu 4}\delta_{\nu 4}. \tag{3}$$

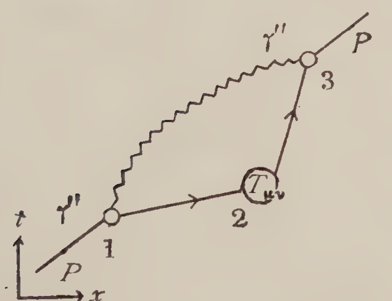

Fig. 1.

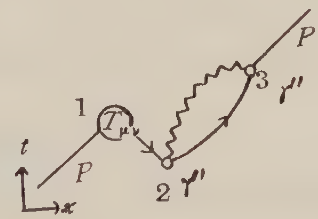

Fig. 2.

a, 2) The case for $t_1 > t_2$ as in Fig. 2. This case just corresponds to the mass type correction in ordinary transition problem.

$$ig^2\int_{-\infty\,(2)}^{1}\int_{(3)} S_+(3,2)\,e^{ipx_3}\gamma^{\cdot\cdot}K_+(3,2)\gamma^{\cdot\cdot}K_+(2,1)\,(-i)\frac{1}{2}\gamma^{\mu}(\overrightarrow{\nabla}-\overleftarrow{\nabla})_{\nu 1}e^{-ipx_1}d\tau_2 d\tau_3$$
$$=i\langle \bar{T}_{44}\rangle\int_{-\infty\,(2)}^{1} e^{ipx_2}K_+(2,1)\;(-i)\frac{1}{2}\gamma^{\mu}(\overrightarrow{\nabla}-\overleftarrow{\nabla})_{\nu 1}e^{-ipx_1}d\tau_2. \tag{4}$$

a, 3) The case for which $T_{\mu\nu}$ occurs in the final knick. This is the transposed of (4).

The case corresponding to $t_1 < t_2$ in Fig. 2. does not exist in perturbation calculation, and is replaced by the following:

$$-\frac{1}{2}ig^2\iint\limits_{(1)(2)} S_+(3,1)e^{ipx_4}\gamma^{\cdot\cdot}K_+(3,2)\gamma^4K_+(2,1)\gamma^{\cdot\cdot}e^{-ipx_1}d\tau_1 d\tau_3\cdot\frac{\bar{p}+m}{2p_4}\cdot$$

$$\cdot e^{ipx_4}(-i)\,\frac{1}{2}\gamma^\mu(\overrightarrow{\nabla}-\overleftarrow{\nabla})_{\nu 4}e^{-ipx_4}+\text{« transposed »} \tag{5}$$

and just corresponds to the contribution of renormalization of state functional in perturbation formulation.

$$\frac{1}{2}\,\frac{g^2}{\pi i}\int\gamma^{\cdot\cdot}(\bar{p}-\bar{k}-m)^{-1}\gamma^4(\bar{p}-\bar{k}-m)^{-1}\gamma^{\cdot\cdot}\,\frac{1}{k^2-\mu^2}d^{(4)}k\cdot(-1)\gamma^\mu p_\nu$$

$$+\ \text{« transposed »}. \tag{6}$$

b) Transition through $T_{\mu\nu}{}^{(\text{field})}$:

$T_{\mu\nu}{}^{(\text{field})}$ has the following form: (in ordinary notation in I):

$$T_{\mu\nu}{}^{\text{field}}=\frac{\partial\phi_{\cdot\cdot}}{\partial X_\mu}\,\frac{\partial\phi_{\cdot\cdot}}{\partial X_\nu}-\frac{1}{2}\delta_{\mu\nu}\left(\frac{\partial\phi_{\cdot\cdot}}{\partial X_\sigma}\,\frac{\partial\phi_{\cdot\cdot}}{\partial X_\sigma}+\mu^2\phi_{\cdot\cdot}\phi_{\cdot\cdot}\right) \tag{7}$$

And the transition diagram is as in Fig. 3. In this case, though (7) contains the differentiation, the perturbation calculation can be translated into Feynman notation directly:

Fig. 3.

$$-\frac{g^2}{4\pi}\iint\limits_{(1)\,(2)}S_+(3,2)(\overleftarrow{\nabla}_\mu\overrightarrow{\nabla}_\nu+\overleftarrow{\nabla}_\nu\overrightarrow{\nabla}_\mu-\delta_{\mu\nu}(\overleftarrow{\nabla}_\sigma\overrightarrow{\nabla}_\sigma-\mu^2))_2$$

$$S_+(2,1)\cdot e^{ipx_2}\gamma^{\cdot\cdot}K_+(3,1)\gamma^{\cdot\cdot}e^{-ipx_1}d\tau_1 d\tau_3$$

$$=\frac{g^2}{\pi i}\int(2k_\mu k_\nu-\delta_{\mu\nu}(k^2-\mu^2))\gamma^{\cdot\cdot}(\bar{p}-\bar{k}-m)^{-1}$$

$$\gamma^{\cdot\cdot}\frac{1}{(k^2-\mu^2)^2}d^{(4)}k. \tag{8}$$

c) Transition through $T_{\mu\nu}{}^{(\text{interaction})}$:

This part gives twice the 4–4 component of stress tensor, and is given by:

$$2g^2\int S_+(2,1)e^{ipx_2}\gamma^{\cdot\cdot}K_+(2,1)\gamma^{\cdot\cdot}e^{-ipx_1}d\tau_2\cdot\delta_{\mu4}\delta_{\nu4}$$

$$=2\frac{g^2}{\pi i}\int\gamma^{\cdot\cdot}(\bar{p}-\bar{k}-m)^{-1}\gamma^{\cdot\cdot}\frac{1}{k^2-\mu^2}d^{(4)}k\cdot\delta_{\mu4}\delta_{\nu4}. \tag{9}$$

In all these expressions, we took Feynman notation throughout, i. e., including dum index.

We denote the sums of above as $\langle T_{\mu\nu}\rangle^{(1)}$, and distinguish from the part due to vacuum polarization $\langle T_{\mu\nu}\rangle^{(2)}$, then this becomes:*

$$\langle T_{\mu\nu}\rangle^{(1)}=-\frac{g^2}{\pi i}\int\gamma^{\cdot\cdot}(\bar{p}-\bar{k}-m)^{-1}(-1)\gamma^\mu(p_\nu-k_\nu)(\bar{p}-\bar{k}-m)^{-1}\gamma^{\cdot\cdot}\frac{1}{k^2-\mu^2}d^{(4)}k$$

---

* The results in I is $\langle T_{\mu\nu}\rangle^{(1)}$ and this is the reason why we asterisked them.

$$+i\langle T_{44}\rangle\int_{-\infty\atop(2)}^{1}e^{ipx_2}K_+(2,1)(-i)\frac{1}{2}\gamma_\mu(\vec{\nabla}-\overleftarrow{\nabla})_{\nu 1}e^{-ipx_1}d\tau_2+\text{ « transposed »}$$

$$+\frac{1}{2}\frac{g^2}{\pi i}\int\gamma^{\cdot\cdot}(\bar{p}-\bar{k}-m)^{-1}\gamma^4(\bar{p}-\bar{k}-m)^{-1}\gamma^{\cdot\cdot}\frac{1}{k^2-\mu^2}d^{(4)}k\cdot\frac{\bar{p}+m}{2p_4}(-1)\gamma^\mu p_\nu+\text{ « transposed »}$$

$$+\frac{g^2}{\pi i}\int(2k_\mu k_\nu-\delta_{\mu\nu}(k^2-\mu^2))\cdot\gamma^{\cdot\cdot}(\bar{p}-\bar{k}-m)^{-1}\gamma^{\cdot\cdot}\frac{1}{(k^2-\mu^2)^2}d^{(4)}k$$

$$+\frac{g^2}{\pi i}\int\gamma^{\cdot\cdot}(\bar{p}-\bar{k}-m)^{-1}\gamma^{\cdot\cdot}\frac{1}{k^2-\mu^2}d^{(4)}k\cdot\delta_{\mu 4}\delta_{\nu 4}. \tag{10}$$

Here, we can easily verify the general relation between self-stress and self-energy stated in I:

$$\langle\bar{T}_{\mu\mu}\rangle^{(1)}=m\frac{\partial}{\partial m}\langle\bar{T}_{44}\rangle+\mu\frac{\partial}{\partial\mu}\langle\bar{T}_{44}\rangle. \tag{11}$$

where

$$\langle\bar{T}_{44}\rangle=\frac{g^2}{\pi i}\int\gamma^{\cdot\cdot}(\bar{p}-\bar{k}-m)^{-1}\gamma^{\cdot\cdot}\frac{1}{k^2-\mu^2}d^{(4)}k,$$

One can show, in the expression (10), the part of first line cancels the third line, and terms of second line do nothing, so (10) becomes:

$$\langle T_{\mu\nu}\rangle^{(1)}=-\frac{g^2}{\pi i}\int\gamma^{\cdot\cdot}(\bar{p}-\bar{k}-m)^{-1}\gamma^\mu k_\nu(\bar{p}-\bar{k}-m)^{-1}\gamma^{\cdot\cdot}\frac{1}{k^2-\mu^2}d^{(4)}k$$

$$+\frac{g^2}{\pi i}\int(2k_\mu k_\nu-\delta_{\mu\nu}(k^2-\mu^2))\cdot\gamma^{\cdot\cdot}(\bar{p}-\bar{k}-m)^{-1}\gamma^{\cdot\cdot}\frac{1}{(k^2-\mu^2)^2}d^{(4)}k$$

$$+\frac{g^2}{\pi i}\int\gamma^{\cdot\cdot}(\bar{p}-\bar{k}-m)^{-1}\gamma^{\cdot\cdot}\frac{1}{k^2-\mu^2}d^{(4)}k\cdot\delta_{\mu 4}\delta_{\nu 4}. \tag{12}$$

Now, the first and second term of this expression can be written as follows:

$$\langle T_{\mu\nu}\rangle^{(1)}=-\frac{g^2}{\pi i}\int\frac{\partial}{\partial k_\mu}\Big[\gamma_{\cdot\cdot}(\bar{p}-\bar{k}-m)^{-1}\gamma^{\cdot\cdot}\Big]\frac{k_\nu}{k^2-\mu^2}d^{(4)}k$$

$$-\frac{g^2}{\pi i}\int\gamma^{\cdot\cdot}(\bar{p}-\bar{k}-m)^{-1}\gamma^{\cdot\cdot}\frac{\partial}{\partial k_\mu}\Big[\frac{k_\nu}{k^2-\mu^2}\Big]d^{(4)}k$$

$$+\frac{g^2}{\pi i}\int\gamma^{\cdot\cdot}(\bar{p}-\bar{k}-m)^{-1}\gamma^{\cdot\cdot}\frac{1}{k^2-\mu^2}d^{(4)}k\cdot\delta_{\mu 4}\delta_{\nu 4}. \tag{13}$$

So, we obtain the following expression for the expectation value of energy-momentum tensor of Fermion interacting with Boson mass μ:

$$\langle T_{\mu\nu}\rangle^{(1)}=-\frac{g^2}{\pi i}\int\frac{\partial}{\partial k_\mu}\Big[\gamma^{\cdot\cdot}(\bar{p}-\bar{k}-m)^{-1}\gamma^{\cdot\cdot}\frac{1}{k^2-\mu^2}k_\nu\Big]d^{(4)}k+\langle\bar{T}_{44}\rangle\delta_{\mu 4}\delta_{\nu 4}. \tag{14}$$

If one sums with index, this is the same as in I Eq. (8.4).
Now, from taking $\mu=\nu=4$, it follows that:

$$-\frac{g^2}{\pi i}\int\frac{\partial}{\partial k_4}\left[\gamma^{\cdot\cdot}(\bar{p}-\bar{k}-m)^{-1}\gamma^{\cdot\cdot}\frac{1}{k^2-\mu^2}k_4\right]d^{(4)}k=0 \tag{15}$$

and from the symmetry consideration for the resting Fermion $(\bar{P}=\bar{P}_0)$:

$$-3\langle \overset{\circ}{T}_{11}\overset{(1)}{\rangle}=-\frac{g^2}{\pi i}\int div\left[\gamma^{\cdot\cdot}(\bar{p}_0-\bar{k}-m)^{-1}\gamma^{\cdot\cdot}\frac{1}{k^2-\mu^2}\vec{k}\right]d^{(4)}k \tag{16}$$

where div means $\partial/\partial k_i$ $(i=1..3)$ and $\vec{k}$ is $\vec{k}(k_1,\ k_2,\ k_3)$.

To evaluate (16), we take example for vector meson field as Boson, then $g=e,\ \gamma^{\cdot\cdot}=i\gamma^\lambda,\ \bar{P}_0=m\gamma^4$:

$$-3\langle T_{11}\overset{(1)}{\rangle}=\frac{e^2}{\pi i}\int div\left[\frac{\gamma^\lambda(m\gamma^4-k_4\gamma^4+\vec{k}\vec{\gamma}+m)\gamma^\lambda\cdot\vec{k}}{(k_4-m-\sqrt{m^2+k^2})(k-m+\sqrt{m^2+k^2})(k_4-\sqrt{\mu^2+k^2})}\right.$$

$$\left.\overline{(k_4+\sqrt{\mu^2+k^2})}\right]\cdot\frac{1}{4\pi^2}dk_4d\vec{k}. \tag{17}$$

According to Feynman, m and μ have small negative imaginary parts and so the poles $k_4=m+\sqrt{m^2+k^2}$, $k_4=\sqrt{\mu^2+k^2}$ lie under the real axis. We can deform the path of integration of k_4 to lower half-plane of complex k_4-space :

$$-3\langle \overset{\circ}{T}_{11}\overset{(1)}{\rangle}=\frac{e^2}{\pi i}\frac{(-2\pi i)}{4\pi^2}\int div\left[\frac{\gamma^\lambda(m\gamma^4-(m+\sqrt{m^2+k^2})\gamma^4+\vec{k}\vec{\gamma}+m)\gamma^\lambda\cdot\vec{k}}{2\sqrt{m^2+k^2}(m+\sqrt{m^2+k^2}-\sqrt{\mu^2+k^2})(m+\sqrt{m^2+k^2}}\right.$$

$$\left.\overline{+\sqrt{\mu^2+k^2})}+\frac{\gamma^\lambda(m\gamma^4-\sqrt{\mu^2+k^2}\gamma^4+\vec{k}\vec{\gamma}+m)\gamma^\lambda\cdot\vec{k}}{2\sqrt{\mu^2+k^2}(-m-\sqrt{m^2+k^2}+\sqrt{\mu^2+k^2})(-m+\sqrt{m^2+k^2}+\sqrt{\mu^2+k^2})}\right]\cdot d\vec{k}$$

$$=-\frac{e^2}{2\pi^2}\int_\sigma\frac{k^2-\sqrt{m^2+k^2}\sqrt{\mu^2+k^2}-3m\sqrt{m^2+k^2}+2\mu^2-m^2}{\sqrt{m^2+k^2}\sqrt{\mu^2+k^2}(m+\sqrt{m^2+k^2}-\sqrt{\mu^2+k^2})(-m^2+(\sqrt{m^2+k^2}+\sqrt{\mu^2+k^2})^2)}$$

$$\times\vec{k}\,d\vec{\sigma}\cdot m. \tag{18}$$

The expression which is obtained by removing div and $\vec{k}$ from (18) is the self-energy expression in ordinary form, and in (18) $d\vec{\sigma}$ is surface element of $k^{(3)}$-space.

If one takes the surface sufficiently apart from the origin of coordinate, then one can expand (18) with k, since it is the distance from the origin to a point on the surface:

$$=-\frac{e^2}{2\pi^2}\int_\sigma\frac{-3mk+\frac{3}{2}\mu^2-\frac{3}{2}m^2+0\left(\frac{1}{k}\right)}{4mk^4}\vec{k}\,d\vec{\sigma}\cdot m$$

$$=-\frac{e^2}{2\pi^2}\int_\sigma\frac{-3}{4}\frac{\vec{k}}{k}\frac{d\vec{\sigma}}{k^2}m+\int_\sigma 0\left(\frac{1}{k^3}\right)d\vec{\sigma}\cdot\frac{\vec{k}}{k}. \tag{19}$$

Since, $\int d\vec{\sigma}\frac{\vec{k}}{k}$ is of order k^2 and the value of k is very large, the last term of (19) vanishes in the limit, and one gets:

$$-\langle \overset{\circ}{\bar{T}}_{11}\rangle^{(1)}=\frac{e^2}{8\pi^2}\int\frac{\vec{k}}{k}\,\frac{d\sigma}{k^2}\cdot m. \tag{20}$$

If one takes σ as sphere of radius k :

$$-\langle \overset{\circ}{\bar{T}}_{11}\rangle^{(1)}=\frac{e^2}{2\pi}m \tag{21}$$

which is the results in perturbation calculation I. It is seen that owing to the limiting process to extend the radius of the k-space to infinity, the value obtained does not depend on the mass of mesons μ. This holds for other cases also, and was quoted in I.

Thus, we obtain the results:

Self-stress of Fermion,

$$-\langle \overset{\circ}{\bar{T}}_{11}\rangle^{(1)}\begin{cases}\text{Due to vector-meson with mass } \mu: & \dfrac{e^2}{2\pi}m & (22)\\ \text{Due to scalar-meson with mass } \mu: & \dfrac{f^2}{4\pi}m & (23)^*\\ \text{Due to pseudoscalar-meson with mass } \mu: & \dfrac{f^2}{12\pi}m. & (24)\end{cases}$$

As one can easily understand, the field-mass independency of the Fermion self-stress comes from the weak divergence of self-energy, and if the self-energy contains quadratic divergences, the self-stress would involve the area of the surface in $k^{(3)}$-space, which of course diverges, and field mass depending term.

d) Polarization term $\langle \bar{T}_{\mu\nu}\rangle^{(2)}$. (Fig. 4) We have neglected this effect in I, since this vanishes for vector and pseudoscalar mesons, but it remains in scalar case, and becomes as follows;

$$\langle \bar{T}_{\mu\nu}\rangle^{(2)}=\frac{ig^2}{\pi\mu^2}\int S_p[(\bar{l}-m)^{-1}\gamma^{\mu}l_{\nu}(\bar{l}-m)^{-1}\gamma^{\cdot\cdot}-(\bar{l}-m)^{-1}\delta_{\mu4}\delta_{\nu4}\gamma^{\cdot\cdot}]\gamma^{\cdot\cdot}d^{(4)}l.$$

The second term is put to cancell the term which is not contained in perturbation calculation which is obtained by following events in Fig. 4. This expression vanishes for $\mu=\nu=4$,

$$\langle \bar{T}_{44}\rangle=0$$

$$-\langle \bar{T}_{11}\rangle^{(2)}=\begin{cases}0 & \text{for vector- and pseudoscalar meson :}\\ -\dfrac{f^2m}{3\pi^2\mu^2}\displaystyle\int\frac{l^2}{(m^2+l^2)^{3/2}}d\vec{t} & \text{for scalar meson :}\end{cases}$$

The inclusion of this term leads us to abandon the assumption of C-meson as cohesive field. In the following discussions we put out of this term.

§ 2. Self-Stress of Boson due to Fermion field.

In this case, also, the transition matrix are classified into three parts:

a) Transition through $T_{\mu\nu}^{(\text{particle})}$:

As in I, the differential operator contained in $T_{\mu\nu}^{(\text{particle})}$ brings super-fluous term in Feynman notation compared with perturbation calculation which is obtained by tracing the events in time (Fig. 5).

a, 1)

$$4\pi f^2 \int_{(1)}\int_{(3)} \frac{e^{ikx_{31}}}{2k_0} S_p[K_+(1,2)(-i)\frac{1}{2}\gamma^{\mu}(\overrightarrow{\Gamma}-\overleftarrow{\Gamma})_{\nu 2}K_+(2,3)\gamma^{\cdot\cdot}K_+(3,1)\gamma^{\cdot\cdot}]d\tau_1 d\tau_3$$

$$-4\pi f^2 \int_{(3)} \frac{e^{ikx_{32}}}{2k_0} S_p[(-i)\frac{1}{2}\gamma^4\gamma^{\mu}\delta_{\nu 4}K_+(2,3)\gamma^{\cdot\cdot}K_+(3,2)\gamma^{\cdot\cdot}]d\tau_3$$

$$-4\pi f^2 \int_{(1)} \frac{e^{ikx_{21}}}{2k_0} S_p[K_+(1,2)(-i)\frac{1}{2}\gamma^{\mu}\gamma^4\delta_{\nu 4}\gamma^{\cdot\cdot}K_+(2,1)\gamma^{\cdot\cdot}]d\tau_1. \tag{25}$$

The last two terms are added to remove the extra term contained in the first expression.

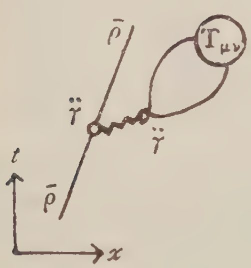

Fig. 4.

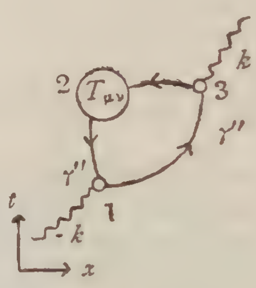

Fig. 5.

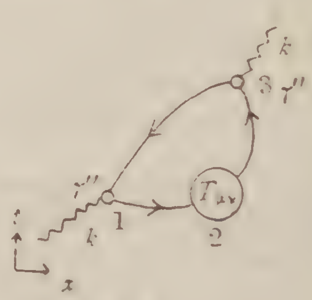

Fig. 6.

a, 2) (Fig. 6)

$$4\pi f^2 \int_{(1)}\int_{(3)} \frac{e^{ikx_{31}}}{2k_0} S_p[K_+(1,3)\gamma^{\cdot\cdot}K_+(3,2)(-i)\frac{1}{2}\gamma^{\mu}(\overrightarrow{\Gamma}-\overrightarrow{\Gamma})_{\nu 2}K_+(2,1)\gamma^{\cdot\cdot}]d\tau_1 d\tau_3$$

$$-4\pi f^2 \int_{(1)} \frac{e^{ikx_{21}}}{2k_0} S_p[K_+(1,2)\gamma^{\cdot\cdot}(-i)\frac{1}{2}\gamma^4\gamma^{\mu}\delta_{\nu 4}K_+(2,1)\gamma^{\cdot\cdot}]d\tau_1$$

$$-4\pi f^2 \int_{(3)} \frac{e^{ikx_{32}}}{2k_0} S_p[K_+(2,3)\gamma^{\cdot\cdot}K_+(3,2)(-i)\frac{1}{2}\gamma^{\mu}\gamma^4\delta_{\nu 4}\gamma^{\cdot\cdot}]d\tau_3. \tag{26}$$

The last two terms are added in the same reason.

Combining these, and translating into l-space, we obtain:

$$\frac{f^2}{\pi i}\frac{1}{2k_0}\int S_p[(\bar{l}-\bar{k}-m)^{-1}(-1)\gamma^{\mu}(l_{\nu}-k_{\nu})(\bar{l}-\bar{k}-m)^{-1}\gamma^{\cdot\cdot}(\bar{l}-m)^{-1}\gamma^{\cdot\cdot}]d^{(4)}l$$

$$+\frac{f^2}{\pi i}\frac{1}{2k_0}\int S_p[(\bar{l}-\bar{k}-m)^{-1}\gamma^{\cdot\cdot}(\bar{l}-m)^{-1}(-1)\gamma^{\mu}l_{\nu}(\bar{l}-m)^{-1}\gamma^{\cdot\cdot}]d^{(4)}l$$

$$+2\frac{f^2}{\pi i}\frac{1}{2k_0}\int S_p[(\bar{l}-\bar{k}-m)^{-1}\gamma^{\cdot\cdot}(\bar{l}-m)^{-1}\gamma^{\cdot\cdot}]d^{(4)}l\cdot\delta_{\mu 4}\delta_{\nu 4}. \tag{27}$$

b) Transition through $T_{\mu\nu}^{(\text{field})}$:

The differential operator in $T_{\mu\nu}^{(\text{field})}$ does no harm on the calculation, and perturbational results are directly translated into Feynman notation. The transition for $t_1 > t_2$ in Fig. 7. just corresponds to perturbational case, if one traces the events in time, and corresponds to mass correction.

$$-if^2\int_{-\infty}^{1}\int_{(2)(3)}\frac{e^{ikx_{31}}}{2k_0}(-\overleftarrow{\nabla}_\mu\overrightarrow{\nabla}_\nu-\overleftarrow{\nabla}_\nu\overrightarrow{\nabla}_\mu+\delta_{\mu\nu}(\overleftarrow{\nabla}_\sigma\overrightarrow{\nabla}_\sigma-\mu^2))_1 S_+(1,2)\cdot$$
$$S_p[K_+(2,3)\gamma^{\cdot\cdot}K_+(2,3)\gamma^{\cdot\cdot}]d\tau_2 d\tau_3$$
$$=-\frac{f^2}{\pi i}\frac{1}{2k_0}\cdot\frac{1}{4k_0^2}\left[\begin{Bmatrix}\vec{k}\\k_0\end{Bmatrix}_\mu\begin{Bmatrix}-\vec{k}\\k_0\end{Bmatrix}_\nu+\begin{Bmatrix}-\vec{k}\\k_0\end{Bmatrix}_\mu\begin{Bmatrix}\vec{k}\\k_0\end{Bmatrix}_\nu-2\delta_{\mu\nu}k_0^2\right]\cdot$$
$$\cdot\int S_p[(\bar{l}-\bar{k}-m)^{-1}\gamma^{\cdot\cdot}(\bar{l}-m)^{-1}\gamma^{\cdot\cdot}]d^{(4)}l. \tag{28}$$

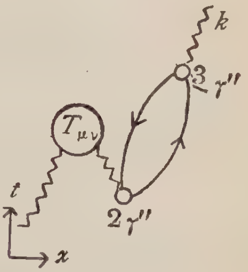

Fig. 7.

And the transposed transition for which the transition takes places in the last knick.

The transition for $t_1 < t_2$ in Fig. 7. does not exist, and in turn there appears the renormalization term:

$$-4\pi f^2\int_{(1)}\int_{(2)}\frac{e^{ikx_{31}}}{2k_0}S_p[K_+(1,2)\gamma^4K_+(2,3)\gamma^{\cdot\cdot}K_+(3,1)\gamma^{\cdot\cdot}]d\tau_1 d\tau_3\ \frac{1}{k_0}\begin{Bmatrix}\vec{k}\\k_0\end{Bmatrix}_\mu\begin{Bmatrix}\vec{k}\\k_0\end{Bmatrix}_\nu$$
$$=-\frac{f^2}{\pi i}\frac{1}{2k_0}\frac{1}{k_0}\begin{Bmatrix}\vec{k}\\k_0\end{Bmatrix}_\mu\begin{Bmatrix}\vec{k}\\k_0\end{Bmatrix}_\nu\cdot\int S_p[(\bar{l}-\bar{k}-m)^{-1}\gamma^4(\bar{l}-\bar{k}-m)^{-1}\gamma^{\cdot\cdot}(\bar{l}-m)^{-1}\gamma^{\cdot\cdot}]d^{(4)}l. \tag{29}$$

c) Transition through $T_{\mu\nu}^{(\text{interaction})}$:

This gives twice 4–4 component:

$$-2\cdot 4\pi f^2 i\int_{(2)}\frac{e^{ikx_{21}}}{2k_0}S_p[K_+(1,2)\gamma^{\cdot\cdot}K_+(2,1)\gamma^{\cdot\cdot}]d\tau_2\cdot\delta_{\mu 4}\delta_{\nu 4}$$
$$=-2\frac{f^2}{\pi i}\frac{1}{2k_0}\int S_p[(\bar{l}-\bar{k}-m)^{-1}\gamma^{\cdot\cdot}(\bar{l}-m)^{-1}\gamma^{\cdot\cdot}]d^{(4)}l\cdot\delta_{\mu 4}\delta_{\nu 4}. \tag{30}$$

Collecting these, we obtain the matrix element of expectation value of the energy-monentum tensor of Boson due to Fermion:

$$\langle T_{\mu\nu}\rangle=\frac{f^2}{\pi i}\frac{1}{2k_0}\int S_p[(\bar{l}-\bar{k}-m)^{-1}(-1)\gamma^\mu(l_\nu-k_\nu)(\bar{l}-\bar{k}-m)^{-1}\gamma^{\cdot\cdot}(\bar{l}-m)^{-1}\gamma^{\cdot\cdot}]d^{(4)}l$$
$$+\frac{f^2}{\pi i}\frac{1}{2k_0}\int S_p[(\bar{l}-\bar{k}-m)^{-1}\gamma^{\cdot\cdot}(\bar{l}-m)^{-1}(-1)\gamma^\mu l_\nu(\bar{l}-m)^{-1}\gamma^{\cdot\cdot}]d^{(4)}l$$
$$-\frac{1}{2k_0^2}\left[-\begin{Bmatrix}\vec{k}\\k_0\end{Bmatrix}_\mu\begin{Bmatrix}-\vec{k}\\k_0\end{Bmatrix}_\nu-\begin{Bmatrix}-\vec{k}\\k_0\end{Bmatrix}_\mu\begin{Bmatrix}\vec{k}\\k_0\end{Bmatrix}_\nu+2\delta_{\mu\nu}k_0^2\right]\cdot$$

$$\left(-\frac{f^2}{\pi i}\right)\frac{1}{2k_0}\int S_p[(\bar{l}-\bar{k}-m)^{-1}\gamma^{\cdot\cdot}(\bar{l}-m)^{-1}\gamma^{\cdot\cdot}]d^{(4)}l$$

$$+\frac{1}{k_0}\left\{\frac{k}{k_0}\right\}_\mu\left\{\frac{k}{k_0}\right\}_\nu\left(-\frac{f^2}{\pi i}\right)\frac{1}{2k_0}\int S_p[(l-k-m)^{-1}\gamma^4(l-k-m)^{-1}\gamma^{\cdot\cdot}(l-m)^{-1}\gamma^{\cdot\cdot}]d^{(4)}l \quad (34)$$

Then, the general relation $\langle\bar{T}_{\mu\mu}\rangle = m\frac{\partial}{\partial m}\langle\bar{T}_{44}\rangle+\mu\frac{\partial}{\partial\mu}\langle\bar{T}_{44}\rangle$ can be proved easily by using the following expressions:

$$\langle\bar{T}_{44}\rangle=-\frac{f^2}{\pi i}\frac{1}{2k_0}\int S_p[(\bar{l}-\bar{k}-m)^{-1}\gamma^{\cdot\cdot}(\bar{l}-m)^{-1}\gamma^{\cdot\cdot}]d^{(4)}l, \qquad (32)$$

$$m\frac{\partial}{\partial m}\langle\bar{T}_{44}\rangle=-\frac{f^2}{\pi i}\frac{1}{2k_0}\int S_p[(\bar{l}-\bar{k}-m)^{-1}m(\bar{l}-\bar{k}-m)^{-1}\gamma^{\cdot\cdot}(\bar{l}-m)\gamma^{\cdot\cdot}]d^{(4)}l$$

$$-\frac{f^2}{\pi i}\frac{1}{2k_0}\int S_p[(\bar{l}-\bar{k}-m)^{-1}\gamma^{\cdot\cdot}(\bar{l}-m)^{-1}m(\bar{l}-m)^{-1}\gamma^{\cdot\cdot}]d^{(4)}l, \qquad (33)$$

$$\mu\frac{\partial}{\partial\mu}\langle\bar{T}_{44}\rangle=\frac{f^2}{\pi i}\frac{1}{2k_0}\frac{\mu^2}{k_0^2}\int S_p[(\bar{l}-\bar{k}-m)^{-1}\gamma^{\cdot\cdot}(\bar{l}-m)^{-1}\gamma^{\cdot\cdot}]d^{(4)}l$$

$$-\frac{f^2}{\pi i}\frac{1}{2k_0}\frac{\mu^2}{k_0}\int S_p[(\bar{l}-\bar{k}-m)^{-1}\gamma^4(\bar{l}-\bar{k}-m)^{-1}\gamma^{\cdot\cdot}(\bar{l}-m)^{-1}\gamma^{\cdot\cdot}]d^{(4)}l. \qquad (34)$$

For the resting meson: $\bar{k}=\bar{k}_0\ (0,\ 0,\ 0,\ \mu\gamma^4)$:

$$\langle\mathring{\bar{T}}_{\mu\nu}\rangle=\frac{f^2}{\pi i}\frac{1}{2\mu}\int S_p[(\bar{l}-\bar{k}_0-m)^{-1}(-1)\gamma^\mu l_\nu(\bar{l}-\bar{k}_0-m)^{-1}\gamma^{\cdot\cdot}(\bar{l}-m)^{-1}\gamma^{\cdot\cdot}]d^{(4)}l$$

$$+\frac{f^2}{\pi i}\frac{1}{2\mu}\int[S_p(\bar{l}-\bar{k}_0-m)^{-1}\gamma^{\cdot\cdot}(\bar{l}-m)^{-1}(-1)\gamma^\mu l_\nu(\bar{l}-m)^{-1}\gamma^{\cdot\cdot}]d^{(4)}l$$

$$-(\delta_{\mu\nu}-\delta_{\mu4}\delta_{\nu4})\left(-\frac{f^2}{\pi i}\right)\frac{1}{2\mu}\int S_p[(\bar{l}-\bar{k}-m)^{-1}\gamma^{\cdot\cdot}(\bar{l}-m)^{-1}\gamma^{\cdot\cdot}]d^{(4)}l. \qquad (35)$$

This expression can also be written as follows:

$$\langle\mathring{\bar{T}}_{\mu\nu}\rangle=\frac{f^2}{\pi i}\frac{1}{2\mu}\int\frac{\partial}{\partial l_\mu}\left\{S_p[(l-k_0-m)^{-1}\gamma^{\cdot\cdot}(\bar{l}-m)^{-1}\gamma_{\cdot\cdot}]l_\nu\right\}\cdot d^{(4)}l$$

$$+\delta_{\mu4}\delta_{\nu4}\langle\mathring{\bar{T}}_{44}\rangle \qquad (36)$$

with $\langle\bar{T}_{44}\rangle$ (32). This is just the same form as (14) for the Fermion self-stress. But, in this Boson case, $\langle\bar{T}_{44}\rangle$ contains the quadratic divergences, and so $\langle\bar{T}_{11}\rangle$ may contain surface-area of $l^{(3)}$-space.

As an example, we exhibit this for the scalar-meson:
Firstly, from the symmetry considerations: ($f=f,\ \gamma^{\cdot\cdot}=1$):

$$-3\langle\bar{T}_{11}\rangle=\frac{f^2}{\pi i}\frac{1}{2\mu}\int div\{S_p[(\bar{l}-k-m)^{-1}(\bar{l}-m)^{-1}]\cdot\vec{l}\}\frac{1}{4\pi^2}dl_4d\vec{l} \qquad (37)$$

$$=\frac{f^2}{2\pi i\mu}\int div\, S_p\left[\frac{((l_4-\mu)\gamma^4-\vec{l}\vec{\gamma}+m)(l_4\gamma^4-\vec{l}\vec{\gamma}+m)\cdot\vec{l}}{(l_4-\mu-\sqrt{m^2+l^2})(l_4-\mu+\sqrt{m^2+l^2})(l_4-\sqrt{m^2+l^2})(l_4+\sqrt{m^2+l^2})}\right]$$

$$\times \frac{1}{4\pi^2} dl_4 d\vec{l} \tag{38}$$

According to Feynman, the poles $\mu+\sqrt{m^2+l^2}$, $\sqrt{m^2+l^2}$ lie under the real axis, and we obtain,

$$= \frac{f^2}{2\pi i\mu}\frac{(-2\pi i)}{4\pi^2}\int div\left[\frac{S_p(\sqrt{m^2+l^2}\gamma^4-\vec{l}\vec{\gamma}+m)((\mu+\sqrt{m^2+l^2})\gamma^4-\vec{l}\vec{\gamma}+m)\cdot\vec{l}}{2\mu\sqrt{m^2+l^2}(\mu+2\sqrt{m^2+l^2})}\right.$$

$$\left.-\frac{S_p((\sqrt{m^2+l^2}-\mu)\gamma^4-\vec{l}\vec{\gamma}+m)(\sqrt{m^2+l^2}\gamma^4-\vec{l}\vec{\gamma}+m)\cdot\vec{l}}{2\mu\sqrt{m^2+l^2}(-\mu+2\sqrt{m^2+l^2})}\right]d\vec{l}.$$

Noting $S_p\gamma^4\gamma^4=4$, $S_p(\vec{l}\vec{\gamma})(\vec{l}\vec{\gamma})=-4l^2$ in Feynman notation,

$$-3\langle \mathring{T}_{11}\rangle = -\frac{f^2}{2\pi^2\mu}\int div\frac{4l^2\cdot\vec{l}}{\sqrt{m^2+l^2}(4(m^2+l^2)-\mu^2)}\cdot d\vec{l}$$

$$= -\frac{f^2}{2\pi^2\mu}\int_\sigma \frac{4l^2}{\sqrt{m^2+l^2}(4(m^2+l^2)-\mu^2)}\vec{l}\,d\vec{\sigma}$$

$$= -\frac{f^2}{2\pi^2\mu}\int\left[1+\left(-\frac{3}{2}m^2+\frac{1}{4}\mu^2\right)\frac{1}{l^2}+0\left(\frac{1}{l^4}\right)\right]\frac{\vec{l}}{l}d\vec{\sigma}. \tag{39}$$

The expression obtained by removing div and $\vec{l}$ from first of this expression is just minus sign of $\langle \mathring{T}_{44}\rangle$ (3.3) of I. The last term can be omitted if one takes the surface to be far apart from the origin.

$$-\langle \mathring{T}_{11}\rangle = -\frac{f^2}{6\pi^2\mu}\int\left[1+\left(-\frac{3}{2}m^2+\frac{1}{4}\mu^2\right)\frac{l}{l^2}\right]\frac{\vec{l}}{l}d\vec{\sigma}. \tag{40}$$

If one takes the surface to be spherical with radius l:

$$-\langle \mathring{T}_{11}\rangle = -\frac{2f^2}{3\pi\mu}\left[l^2+\left(-\frac{3}{2}m^2+\frac{1}{4}\mu^2\right)\right]_{l\to\infty}. \tag{41}$$

The result for the pseudoscalar meson is as follows:

$$-\langle \mathring{T}_{11}\rangle = -\frac{2f^2}{3\pi\mu}\left[l^2+\left(-\frac{1}{2}m^2+\frac{1}{4}\mu^2\right)\right]_{l\to\infty}. \tag{42}$$

Thus, in this case, the self-stress contains mass of Fermion and surface area of l-space, which is due to the fact that the divergence of the self-energy is quadratic, as one can easily understand.

The value thus obtained is dependent on the form of the l-space taken, and the finite value is not unique, but for the resting meson, it will be natural to assume the spherical $l^{(3)}$-space, since there may be no reason to be non-spherical due to any unsymmetry.

§ 3. A physical Interpretation of Self-Stress of Fermion.

The expression (14) for the additional stress of Fermion can be interpreted

physically in the analogous way as Poincaré tensor[1)] in classical electrodynamics. To make the meaning clearer, we treat the problem by introducing sufficiently strong cut-off factor $g(k^2)$ in the emission and absorption point, (where $\gamma^{\cdot\cdot}$ presents), which, of course, may be brought to 1 after the calculations. This has the consequence that the interaction Hamilton density becomes:

$$g\psi^\dagger(X)\gamma^{\cdot\cdot}\psi(X)\int G(X-X')\phi_{\cdot\cdot}(X')\,dx'\ ;\ \ G(X)=\frac{1}{(2\pi)^4}\int g(k^2)e^{-ikx}d^{(4)}k. \qquad (43)$$

And, self-stress becomes as follows:

$$\langle \mathring{T}_{44}\rangle=\frac{g^2}{\pi i}\int\gamma^{\cdot\cdot}(\bar{p}_0-\bar{k}-m)^{-1}\gamma^{\cdot\cdot}\frac{1}{k^2-\mu^2}\{g(k^2)\}^2d^{(4)}k \qquad (44)$$

$$\langle \mathring{T}_{11}\rangle=-\frac{g^2}{\pi i}\int\frac{\partial}{\partial k_1}\Big[\gamma^{\cdot\cdot}(\bar{p}_0-\bar{k}-m)^{-1}\gamma^{\cdot\cdot}\frac{1}{k^2-\mu^2}k_1\Big]\{g(k^2)\}^2d^{(4)}k. \qquad (45)$$

Now, the system with interaction Hamilton density (43) gives the force density acting on the charge element with center at 0 in 1-direction:

$$g\psi^\dagger(0)\gamma^{\cdot\cdot}\psi(0)\cdot G(Y)\frac{\partial}{\partial Y_1}\phi_{\cdot\cdot}(Y). \qquad (46)$$

The expectation value of this for the state with one electron becomes:

$$g^2G(Y)\frac{\partial}{\partial Y_1}\int G(X-Z)\,S_+(Z,Y)\gamma^{\cdot\cdot}K_+(X,0)\gamma^{\cdot\cdot}e^{ip_0X}d^{(4)}Xd^{(4)}Z$$

$$+g^2G(Y)\frac{\partial}{\partial Y_1}\int G(X-Z)\,S_+(Y,Z)\gamma^{\cdot\cdot}K_+(0;X)\gamma^{\cdot\cdot}e^{-ip_0X}d^{(4)}Xd^{(4)}Z \qquad (47)$$

$$=g^2G(Y)\frac{\partial}{\partial Y_1}\int\frac{1}{(2\pi)^4\pi i}g(k'^2)e^{-ik'(X-Z)}\Big\{\frac{e^{-ik(Z-X)}}{k^2-\mu^2}\gamma^{\cdot\cdot}(\bar{l}-m)^{-1}\gamma^{\cdot\cdot}e^{-ilX+ip_0X}$$

$$+\frac{e^{ik(Z-Y)}}{k^2-\mu^2}\gamma^{\cdot\cdot}(\bar{l}-m)^{-1}\gamma^{\cdot\cdot}e^{ilX-ip_0X}\Big\}d^{(4)}X\,d^{(4)}Z\,d^{(4)}k\,d^{(4)}k'\,d^{(4)}l \qquad (48)$$

$$=G(Y)\cdot\frac{\partial}{\partial Y_1}\cdot\frac{g^2}{\pi i}\int g(k^2)\frac{e^{ikY}+e^{-ikY}}{k^2-\mu^2}\gamma^{\cdot\cdot}(\bar{p}_0-\bar{k}-m)^{-1}\gamma^{\cdot\cdot}d^{(4)}k=F_1(Y). \qquad (49)$$

And, if one imagines that when the particle is free, it is a "mathematical point", but when it emits or absorbs the field, it spreads over $G(X)$ acted by the constant force at the end point given by (49). Then, the work done by this force during the expansion in 1-direction is given by the following:

$$\int Y_1F_1(Y)d^{(4)}Y=\frac{2g^2}{\pi i}\iint g(k^2)\,g(k'^2)\,\frac{1}{k^2-\mu^2}\gamma^{\cdot\cdot}(\bar{p}_0-\bar{k}-m)^{-1}\gamma^{\cdot\cdot}\cdot$$

$$k_1\frac{\partial}{\partial k_1}\delta^{(4)}(k-k')\,d^{(4)}k\,d^{(4)}k' \qquad (50)$$

$$=-\frac{g_2}{\pi i}\int\frac{\partial}{\partial k_1}[\gamma^{\cdot\cdot}(\bar{p}_0-\bar{k}-m)^{-1}\gamma^{\cdot\cdot}\frac{1}{k^2-\mu^2}k_1\{g(k^2)\}^2]d^{(4)}_k$$

$$-\frac{g^2}{\pi i}\int\frac{\partial}{\partial k_1}[\gamma^{\cdot\cdot}(\bar{p}_0-\bar{k}-m)^{-1}\gamma^{\cdot\cdot}\frac{1}{k^2-\mu^2}k_1]\{g(k^2)\}^2d^{(4)}k. \qquad (51)$$

The first term of this expression vanishes owing to the sufficiently strong cut-off factor $g(k^2)$ (see (19) or §4,b)), and the second term is the same as (45).

And at this point one can take the limit $g(k^2)\rightarrow 1$ and the expression (51) becomes to the original expression (16).

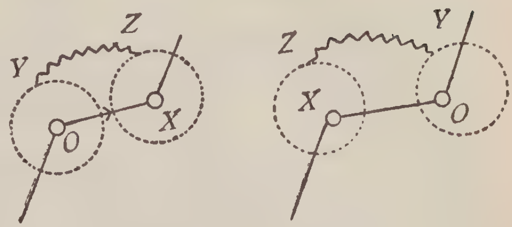

Fig. 8.

So that, one can imagine the mechanism of the origin of self-stress of Fermion as follows: the particle is a mathematical point when free, and when it emits or absorbs the field it spreads over $G(X)$, and during this spreading the charge element is worked by the constant force due to the field just equal to the force acted at the end point. This is the analogy of interpretation of self-stress in classical electrodynamics due to Poincare.

The fact that for point electron, the self-stress remains finite indicates that the sigularity of expanding force near the electron does finite work to bring the "mathematical" point charge to the limit of extended charge.

§ 4. Convergence Conditions and Stabilization Problems.

From above results, it is easily seen that the self-stress is intimately connected with self-energy through simple relations. We now want to discuss the relation between convergence of self-energy and of vanishment of self-stress in three steps, i. e., from the standpoint of extended model of particle, the realistic subtraction procedure such as Pais or Sakata, and from the purely formalistic stand point and semi-formalistic standpoint.

a) Extended models. If one makes a particle extend in some meaning, it will result a cut-off factor $g(k^2)=\sqrt{C(k^2)}$ for emission and absorption point, so that, the energy-momentum becomes:

For Fermion:*

$$\langle \overset{\circ}{T}_{\mu\nu}\rangle = -\frac{g^2}{\pi i}\int \frac{\partial}{\partial k_\mu}\Big[\gamma^{\cdot\cdot}(\bar{p}_0-\bar{k}-m)^{-1}\gamma^{\cdot\cdot}\frac{1}{k^2-\mu^2}k_\nu\Big]C(k^2)\,d^{(4)}k$$

$$+\frac{g^2}{\pi i}\int \gamma^{\cdot\cdot}(\bar{p}_0-\bar{k}-m)^{-1}\gamma^{\cdot\cdot}\frac{1}{k^2-\mu^2}C(k^2)\,d^{(4)}k\cdot\delta_{\mu 4}\delta_{\nu 4}. \tag{52}$$

For Boson:

$$\langle \overset{\circ}{T}_{\mu\nu}\rangle = \frac{f^2}{\pi i}\,\frac{1}{2\mu}\int \frac{\partial}{\partial l_\mu}\Big\{S_p[(\bar{l}-\bar{k}_0-m)^{-1}\gamma^{\cdot\cdot}(\bar{l}-m)^{-1}\gamma^{\cdot\cdot}]\cdot l_\nu\Big\}\cdot C(l^2)\,d^{(4)}l$$

$$-\frac{f^2}{\pi i}\,\frac{1}{2\mu}\int S_p[(\bar{l}-\bar{k}-m)^{-1}\gamma^{\cdot\cdot}(\bar{l}-m)^{-1}\gamma^{\cdot\cdot}]C(l^2)\,d^{(4)}l\cdot\delta_{\mu 4}\delta_{\nu 4}. \tag{53}$$

The 4–4 component of these surely converges by choosing the function C so as to

diminish the singularity. But, the self-stress also remains finite and not becomes to zero. This can be seen from the following considerations:

$$-\frac{g^2}{\pi i}\int\frac{\partial}{\partial k_1}[\gamma\cdot\cdot(\bar{p}-\bar{k}-m)^{-1}\gamma\cdot\cdot\frac{1}{k^2-\mu^2}k_1\cdot C(k^2)]d^{(4)}k \qquad \text{(Fermion)} \qquad (54)$$

$$\frac{f^2}{\pi i}\frac{1}{2\mu}\int\frac{\partial}{\partial l}\{S_p[(\bar{l}-\bar{k}-m)^{-1}\gamma\cdot\cdot(\bar{l}-m)^{-1}\gamma\cdot\cdot]l_1\cdot C(l^2)\}d^{(4)}l \qquad \text{(Boson)} \qquad (55)$$

surely vanishes if C is chosen so as to make converge the self-energy (see, b) But there remains:

$$\frac{g^2}{\pi i}\int\gamma\cdot\cdot(\bar{p}-\bar{k}-m)^{-1}\gamma\cdot\cdot\frac{1}{k^2-\mu^2}k_1\frac{\partial}{\partial k_1}C(k^2)d^{(4)}k \qquad \text{(Fermi)} \qquad (56)$$

$$-\frac{f^2}{\pi i}\frac{1}{2\mu}\int S_p[(\bar{l}-\bar{k}-m)^{-1}\gamma\cdot\cdot(\bar{l}-m)^{-1}\gamma\cdot\cdot]l_1\cdot\frac{\partial}{\partial l_1}C(l^2)\cdot d^{(4)}l \qquad \text{(Bose)} \qquad (57)$$

which does not vanish in general and remains finite, and can be interpreted to be the work done by the field during the expansion of particle (§ 3). So that, *in the extended model*, generally, *the convergence of self-energy does not secure the stability of the particle.*

b) Realistic subtraction procedures.

In the system of particle and several fields, the self-stress is given by the following (see (16) and (37)): Write the self-energy in the K-space of interacting fields (the contribution from each fields, we distinguish by i)

$$\langle\overset{\circ}{T}_{44}\rangle=\int\sum_i W_i(K)d\vec{K}. \qquad (58)$$

Then self-stress is led by the following:

$$\langle\overset{\circ}{T}_{11}\rangle=-\frac{1}{3}\int div_K[\sum_i W_i(K)\cdot\vec{K}]\,d\vec{K} \qquad (59)$$

$$=-\frac{1}{3}\int\sum_i W_i(K)\cdot\vec{K}d\vec{\sigma} \qquad (60)$$

Using this fact we can survey the relation between convergence condition of self-energy and vanishing condition of self stress (stabilization condition).

The convergence of self-energy (58) is maintained if the mixture $\sum W_i(K)$ has asymptotic form:

$$\sum_i W_i(K)\sim\frac{1}{K^{3+\alpha}} \quad (\alpha>0) \qquad (61) \text{ (convergence condition)}$$

for sufficiently large K.

On the other hand, the stabilization condition produces the restriction on $\sum W_i(K)$ from (60), and if one takes σ sufficiently apart from the origin, this produces:

$$\langle\overset{\circ}{T}_{11}\rangle=-\frac{4\pi}{3}\sum_i W_i(K)\cdot K^3\Big|_{K\to\infty}=0$$

$$\sum W_i(K) \sim \frac{1}{K^{3+a}} \quad (a>0) \qquad (62) \text{ (stabilization condition)}$$

i. e., the convergence condition (61) is the same for stabi'ization condition. So that, in general, the mixture which makes converge the self-energy of particle (Fermion* or Boson) also stabilizes the particle.

Thus, *in the realistic theory, the convergence of self-energy automatically leads to the stabilization of particle, and vice versa.** This is because the force density acting in the expansion of particle is given by (49) at distance Y apart from the center of the particle, and which is to be multiplied by Y_1 to produce the self-stress. The convergence of self-energy indicates that the force density near the particle is regular and so product vanishes according to $Y_1\delta^{(4)}(Y)=0$, since integral on the right-hand side of (49) has the same singularity as self-energy near the center of particle ($Y=0$).

The Fermion* can thus be described as relativistic stabile particle in this procedure, but as was shown in the paragraph 1, I, the Boson cannot. The remained finite self-enetgy cannot be renormolized to original mass. So if one wants to describe the Boson with mass $\mu+\delta\mu$ along the idea of "renormalization of mass", the convergence condition is very strong than (61), i. e., $\sum W_i(K)=0$.

c) Formalistic subtraction procedure and semi-formalistic procedure.

In the formalistic theory developed by Pauli and Villars, the added redundant fields have not their own energy-momentum tensors, so that self-stress cannot be expressed as (16) or (37), but in the form subtracted from them the term of the sum of type (8) or (27) due to the contribution of the energy-momentum tensors of redandant fields. And this subtracted term cannot be made vanish with any regularization. So that, it will be preferable to simplify the considerations, to add to the subsidiary field their own energy momentum tensors and so consider them as real fields. But this will bring the difficulty of negative energy fields, so that after all to bring their mass to infinity and to remove the effects in real transition is necessary.

This procedure is realistic, since the redundant fields have their own energy-momentum tensors, but formalistic since the effects of redundant fields only appear in the virtual process. We may call this as semi-formalistic procedure.

The situation between convergence condition and stabilization condition is slightly changed in this theory. For simplicity, we consider a mixture of redundant fields with mass labelled by $i(=1, \cdots, N)$ and which interacts in the same way as original field.

Firstly, for Fermion, the 4–4 component becomes as follows:

$$\langle \overset{\circ}{T}_{44} \rangle = \frac{g^2}{\pi i} \int \gamma^{\cdot\cdot} (\bar{p}_0 - \bar{k} - m)^{-1} \gamma^{\cdot\cdot} \sum_{i=0}^{N} \frac{\eta_i}{k^2-\mu_i^2} d^{(4)}k. \quad (\mu_0=\mu)(\gamma_0=1) \qquad (63)$$

And, as is well known, the convergence condition is as follows:

$$\sum_{i=0}^{N} \eta_i = 0\,; \quad \sum_{i=0}^{N} \eta_i \log \mu_i = 0. \qquad \text{(convergence condition for Fermion)} \quad (64)$$

On the other hand, 1–1 component is given by:*

$$-\langle \mathring{\bar{T}}_{11}\rangle=-\frac{g^2}{\pi i}\int\frac{\partial}{\partial k_1}[\gamma^{\cdot\cdot}(p_0-k-m)^{-1}\gamma^{\cdot\cdot}\sum_{i=0}^{N}\frac{\eta_i}{k^2-\mu_i^2}\cdot k_1]d^{(4)}k \tag{65}$$

which becomes as follows: (see last argument of § 1):

$$=\sum_{i=0}^{N}\eta_i\cdot\text{Const.} \tag{66}$$

Thus, the stabilization condition becomes:

$$\sum_{i=0}^{N}\eta_i=0. \qquad (67)\ \text{(stabilization condition for Fermion).}$$

Thus, the first condition of convergence condition (64) is sufficient to stabilize the Fermion.* The second of convergence condition (64) is related to the logarithmic divergence of self-energy, and since $\langle\bar{T}_{11}\rangle$ does not contain this divergence as discussed in the last of § 1, so that the first is sufficient. Secondly, for Boson the 4–4 component is,

$$\langle\mathring{\bar{T}}_{44}\rangle=-\frac{f^2}{\pi i}\frac{1}{2\mu}\int\sum_{i=0}^{N}\eta_i S_p[(\bar{l}-\bar{k}_0-m_i)^{-1}\gamma^{\cdot\cdot}(\bar{l}-m_i)^{-1}\gamma^{\cdot\cdot}]d^{(4)}l \tag{68}$$

which is made convergent by the conditions:

$$\sum_{i=0}^{N}\eta_i=0,\qquad\sum_{i=0}^{N}\eta_i m_i^2=0,$$

$$\sum_{i=0}^{N}\eta_i\log m_i=0,\quad\sum_{i=0}^{N}\eta_i m_i^2\log m_i=0. \qquad (69)\ \text{(convergence condition for Boson)}$$

And

$$-\langle\mathring{\bar{T}}_{11}\rangle=\frac{f^2}{\pi i}\frac{1}{2\mu}\int\frac{\partial}{\partial l_1}\left\{\sum_{i=0}^{N}\eta_i S_p[(\bar{l}-\bar{k}_0-m_i)^{-1}\gamma^{\cdot\cdot}(\bar{l}-m_i)^{-1}\gamma^{\cdot\cdot}]\cdot l_1\right\}d^{(4)}l. \tag{70}$$

This is of the form: (see last of § 2):

$$=\sum_{i=0}^{N}\eta_i(al^2+b\mu^2+Cm_i^2)_{l\to\infty}. \tag{71}$$

And the stabilization condition becomes:

$$\sum_{i=0}^{N}\eta_i=0,\quad\sum\eta_i m_i^2=0. \qquad (82)\ \text{(stabilization condition for Boson)}$$

Thus, the above consideration indicates that *the stabilization of particle in the semi-formalistic procedure is insufficient to make converge the self energy. The stabilization seems to be related to the removal of ambiguity in the calculations.**

This semi-formalistic procedure is as powerful as that of purely formalistic regularization for the process in which the energy-momentum tensor of redundant field does nothing, (for example, self-energy or other transition in external field), the difference will apprear in which the energy-momentum tensor of redundant field does something as in the self-stress.

Though, the convergence condition (64) or (68) can make self-energy converge and self-stress vanish, the Boson with mass μ has finite self-energy so that Boson with mass $\mu+\delta\mu$ cannot be described in renormalization procedure of mass, as before.

§ 5. Conclusions.

It was shown that the self-stress of Boson and Fermion* can be expressed as the two-dimensional surface integral in k- or $l^{(3)}$-space. And if one takes the surface to be sphere, since the radius of the surface is infinity, the surface integral reduces to the ratio of rational functions of k or l and so self-stress does not contain logarithmic divergence with only one exception,* its divergence is only multiple of surface area of this sphere (which is of course divergent). The self-stress of Fermion treated in this paper is independent of mass of fields, that of Boson contains surface area and mass depending term. The difference comes from the order of divergence on the surface, and is related to the quadratic divergence or logarithmic divergence of self-energy.

A model for Fermion to interpret the origin of self-stress is given which shows that in quantum theory also the model such as Poincare in classical electrodynamics is possible.

The semi-formalistic procedure to remove the self-stres was shown in the last paragraph.

Although it is possible to obtain the vanishing self-stress for Boson and Fermion in some procedure, the Boson cannot be described in renormalization technik as emphasized in several places. The vanishment of self-energy and self-stress simultaneously is necessary to do so, and this leads to very strong convergence conditions. These points will be discussed later.

In conclution, I say much thanks to Prof. M. Kobayasi for his kind interest taken on this work.

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Theory of Color Centers in Ionic Crystals. I*.

—*Electronic States of F-Centers in Alkali Halides.*—

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The potential field in the vicinity of the F-center has been critically discussed and on this ground a simple model of rigid lattice has been introduced. Two methods of approach—the continuum model and large molecule model—have been employed. By use of the variation principle, the electronic states of F-center have been calculated for several models proposed here. The results of calculation show that the calculated energy approaches to the observed ones as the precision of our model increases. The method how to formulate the effect of polarization has been considered, and the relation of our model to that of Tibbs has been discussed.

§ 1. Introduction.

Both experimental and theoretical investigations of recent years on the colored absorption bands in ionic crystals seem to confirm fairly completely the defect model of color centers.

This model assumes the existence of electrons in quasi atomic states trapped in the vicinity of the lattice imperfection of ionic crystals, and the colored bands are due to the electronic transitions between these states. Main results of these problems have excellently been summalized in Mott-Gurney's book[1] and (more recent surveys being given) in Seitz's article.[2]

Among these colored bands, the so-called F-bands of alkali halides play the same role in the spectroscopy of ionic crystals as the alkali metal spectra do in the case of atomic system.

The absorption spectra of the F-bands have not sharp line structure like those of atoms but have bell-shaped one whose peak wave length and width vary with the temperature. In the accuracy in which we have no need to consider the width and its temperature dependence, the peak wave length λ_{max} obeys to a simple law. Indeed, Mollwo[3] first indicated that the peak is proportional to the square of interionic distances. In Ivey's[4] recent research, an experimental formula :

$$\lambda_{max}=703\ d^{1.84} \tag{1.1}$$

has been established. In this formula, λ_{max} and d denote the wave length and

* Read Mar. 28th, 1949 at the annual meeting of Phys. Soc. Japan.

corresponding interionic distance respectively, expressed in Å unit. The proportionality constant is common to all alkali halide crystals with the lattice structure of NaCl type.

The calculation of electronic states under consideration has been carried out by Tibbs[5] for the first time. More recently, Kubo[6] and Nagamiya[7] treated the matter using the spherical symmetric potential field while Muto[8] proposed to treat the matter in a molecular theoretical way. Further, in his second paper, Muto discussed also the temperature dependence of the band. The purpose of the present paper is to consider the electrostatic field in the vicinity of the ion vacancy systematically and models the lattice with the imperfection as a large symmetric molecule whose center coincides with the lattice defect in question. In the Part I of this report the calculation of electronic energy states has been carried out and the calculated values have been compared with Ivey's experimental formula. As to the temperature dependence of the band, the results of observations will be referred to Mollowo s paper and the theoretical interpretation of which will be touched in Part II of this report.

§ 2. The field in the vicinity of the ion vacancy.

In the vicinity of the negative ion vacancy, there grows a potential field which attracts the electron. The energy states of this trapped electron are determined by solving the Schroedinger equation for one body problem in this potential field. In this section, we consider the field near the vacancy more in detail.

As to the treatment of the perfect crystal, we choose the simplest and reliable theory of ionic crystal of Born[9] with appropriate modification. There may occur the displacement and change of the charge distribution if any sort of lattice defect is introduced. For instance, the ions near the vacancy may displace from the normal positions and electron cloud around them yields in polarized state deviating from its initial distribution. The polarization of the lattice consists of these two effects. The lattice will be called a rigid one if these effects may be neglected properly. In the zeroth order approximation, we consider the rigid lattice even if some sort of vacancy is introduced. This is the idealized picture of F-center employed here.

The potential field acting on an electron in a rigid lattice is as follows. If we choose the vacant lattice point as the origin of the coordinate system, it is given by

$$V(\vec{r}) = \sum_{all}{}' V_i(\vec{r} - \vec{R}_i) \tag{2.1}$$

where $\vec{r}$ denotes the position vector of the electron and $\vec{R}_i$ the position vector of the i-th lattice point. $\sum_{all}'$ means that the summation is to be extended to all lattice points with the exception of the origin. Considering that the potential due

to the lattice points near the origin are only effective, (2.1) may be approximated by

$$V(\vec{r}) = \sum_{eff.} V_i(\vec{r}-\vec{R}_i\,,\ z_i) + \int_{all\text{-}eff.} V(\vec{r}-\vec{R})d\tau_{\vec{R}}. \tag{2.2}$$

In the summation of the first term, we take into account of the field due to the ions one by one starting from the nearest neighbors to further points as we proceed to higher order approximations. This corresponds to the picture of the large molecule. If we specially consider the case where our field may be approximated only in terms of the second part of (2.2), it corresponds to the continuum model. On the contrary, if we approximately use the first term only, by chosing the parameter z_i suitably, this corresponds to the model of the large molecule. Some simpler cases are schematically shown in Fig. 1. Stating the matter more in detail, they are (See Fig. 1),

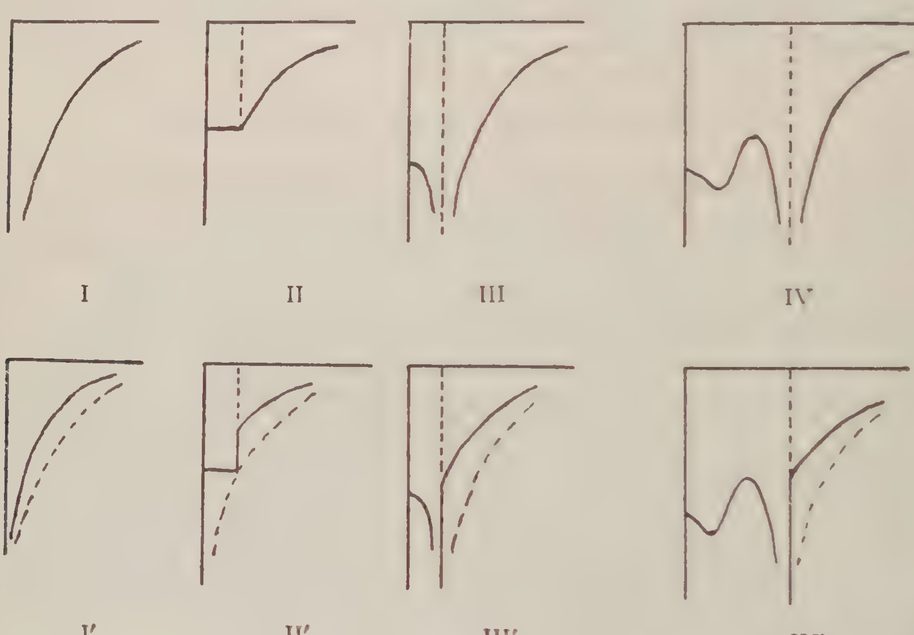

Fig. 1. Schematic representation of potential field.

Case I) The crystal lattice is considered as a perfect continuum. The effect of the negative ion vacancy is expressed by a single point charge located at the origin, the field is exactly equal to that of a hydrogen atom and any characteristics of the material are not introduced in this model at all.

Case II) This corresponds to the potential field due to a uniform charge distribution over a sphere of radius d buried in a continuum. The total charge is equal to 1. Here the characteristic of the material is represented by the radius d.

Case III) This correspondens to the simplest model of the large molecule where only the potential due to the six nearest neighbor positive ions are taken into account. Considering that the potential should tend to $1/r$ as r tends to infinity, we take the value of $z_1=1/6$.

Case IV) This corresponds to the higher order approximation of the large molecule model, where the potential due to twenty six positive and negative ions represented in Fig. 2 are taken into account. As to the value of the effective charge z_i, we choose the value $z_1=1/2$ as for six nearest neighbors $z_2=1/4$ for the eight second neighbors and $z_3=1/8$ as for the eight neighbors. These values for z_i are the same numerical values as they were chosen in the simplified method for evaluating the Madelung constant suggested by Evjen[10] which has been proved so successful. It satisfies the asymptotic behavior $1/r$ as $r \to \infty$ as well as its behavior near the origin that it should tend to Madelung's potential. Thus this large molecule model seems to be a good one.

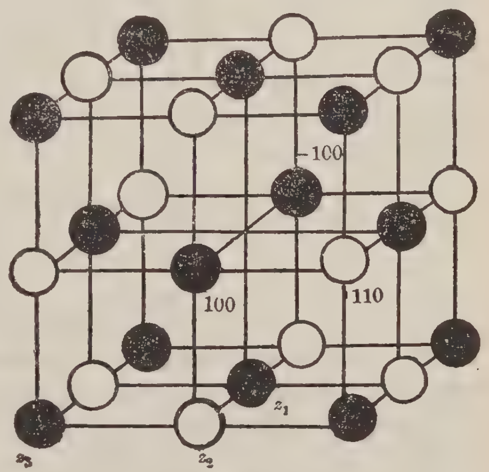

Fig. 2 NaCl, lattice with a negative ion vacancy.

The primed figures in Fig. 1 are the plots of the curves where the effect of polarization is also taken into account besides the terms whose plots are given in the respective upper figures.

§ 3. Solution of the Schroedinger Equation.

The electronic energy states are determined by solving the Schroedinger equation

$$H\Psi = E\Psi \tag{3.1}$$

where

$$H = -\Delta/2 + V. \tag{3.2}$$

In the Hamiltonian H, we choose a certain form of potentials stated in the preceding section and seek for the approximate eigen-values E by the variation principle. Here we must determine the form of the variation functions. We assume the simplest hydrogen like function in the continuum model A (Case I and II). On the other hand, in the case of the large molecule model B (Case III and IV), we select a function which is given by the linear combinations of wave functions whose centers lie in the six nearest lattice points. Each case will be illustrated separately.

A) Continuum model

It seems needless to say more for the case I. As to the case II, we assume for the function of the ground state the form

$$\Psi=(\beta^3/\pi)^{1/2}\ \exp\ (-\beta r) \tag{3.3}$$

and try to minimize the energy value E by varying the parameter β. Let the corresponding value be denoted by β_0, then we get as the energy value $E^{(1)}$ for ground state

$$E^{(1)}=\beta_0^2/2-(1/d)[1-(1+\beta_0 d)\ \exp\ (-2\beta_0 d)]. \tag{3.4}$$

The wave function corresponding to the first excited state is determined as the hydrogen-like $2p$ function which is orthogonal to (3.3). Corresponding eigen-value is given by

$$E^{(2)}=\beta_0^2/8-(1/d)[1-\{(1/24)\ \exp\ (-\beta_0 d)\}\times \\ \{(\beta_0 d)^3+6(\beta_0 d)^2+18(\beta_0 d)+24\}]. \tag{3.5}$$

B) Large Molecule model

The employed method for determining the eigen-value is the same as the method of molecular orbit used by Muto l.c..[5] The chosen potential function and the corresponding variation function are as follows:

$$V(r)=\sum_{eff.} V_i(\vec{r}-\vec{R}_i\ ;\ z_i), \tag{3.6}$$

$$\Psi(r)=\sum_{i=1}^{6} c_i\psi(\vec{r}-\vec{R}_i) \tag{3.7}$$

where $\psi(\vec{r}-\vec{R}_i)$ is a spherical symmetric function with respect to the i-th lattice point. By use of the variation principle we get a secular equation of the order six, which is factorized into three parts owing to the symmetry. The ground state is characterized by its simple root $E^{(1)}$ with the symmetry A_{1g}. Further we have triply degenerate state $E^{(2)}$ whose symmetry is given by F_{1u}. There remains doubly degenerate state $E^{(3)}$ whose symmetry is given by E_g.

The transition between the ground state and last stated doubly degenerate state is forbidden as the ordinary group theoretical consideration teaches us. Thus we need to consider the state A_{1g} and F_{1u} only. In this manner all energy states are obtained for the respective case.

Now the total energy of the lattice is given by

$$W^{(2)}=V_{LL}^{(2)}+E^{(2)},\quad W^{(1)}=V_{LL}^{(1)}+E^{(1)}$$

where V_{LL} denotes the electro-static energy due to interionic interaction. If we consider the effect of polarization it may change from state to state. But the difference between them may rather be small. Thus the wave number of the

F-absorption band is approximately given by the difference between the two electronic terms

$$F = E^{(2)} - E^{(1)}. \tag{3.8}$$

§ 4. Calculations and results.

In the continuum model, the peak wave numbers (3.8) for the case II have been calculated as the difference of term values (3.4) and (3.5). The chosen value d for respective element taken from experiment is given in Table 1. Here, we remark once more that the material characteristic is introduced only through the ionic distance. The results are tabulated in the row II of Table 2.

The actual calculations in the case of the large molecule model stated in the preceding section have been carried out for lithium halides. As for the wave function in (3.7), we employed ψ, $2S$ Hartree function of Li-atom calculated by Obi.[11] It is expressed in a.u. as

	F	Cl	Br	I
Li	3.92	4.86	5.20	5.67
Na	4.36	5.32	5.64	6.11
K	5.05	5.94	6.21	6.68
Rb	5.33	6.18	6.48	6.92

Table 1. Inter-ionic distance in atomic unit.

		F	Cl	Br	I
Li	*III*	0.217	0.170	0.155	0.135
	IV	0.157	0.127	0.116	0.105
	E	0.180	0.115	0.101	0.086
	II	0.091	0.076	0.073	0.065
	I′	0.122	0.067	0.063	0.053
Na	*E*	0.139	0.097	0.087	0.075
	II	0.084	0.072	0.064	0.060
	I′	0.143	0.086	0.079	0.068
K	*E*	0.106	0.079	0.072	0.064
	II	0.074	0.060	0.058	0.052
	I′	0.143	0.105	0.095	0.083
Rb	*E*	0.096	0.073	0.067	
	II	0.070	0.058	0.054	
	I′	0.133	0.103	0.091	

Table 2. Peak of F-band $h\nu$ in atomic unit.

$$\psi(t) = N[\exp(-1.94t) - 0.268t \exp(-0.66t)] \tag{4.1}$$

where N is the normalization constant. As for the potential function (3.6), we employ

$$V_1(t, z_1) = (1/t)[-2 - (2 + 5.4\,t)\exp(-5.40t)] \tag{4.2}$$

as the potential due to Li-ion. This is calculated using Obi's result. In the actual calculation, the effect of only six nearest neighbors have been taken into account, and for the outer, their effect are substituted by their effective point charges:

$$V_i(t, z_i) = \pm z_i/t. \tag{4.3}$$

If the form of the wave function and the potential once fixed, the calculation of the matrix elements of our secular equation may be reduced to those of

molecular integrals familiar to the chemical binding. Its details will be given in the Mathematical Appendix and in this section we briefly sketch our method. Some rather cumbersome integrals encountered are the two- and three-center integrals familiar to the molecular binding. (See Fig. 3). Their integrands are the products of $(1/t)$ exp $(-at)$ with the wave function of form

$$tm_i \exp(-b_i t) \quad m_i=0,\ 1,\ 2\cdots\cdots$$

For the evaluation of these integrals, Coulson[12] has deviced an ingenious method in his treatise on molecular integrals. Most of our integrals have been evaluated according to his scheme and some others along the similar lines. The three center integrals have also been reduced by Coulson to the form of infinite series, but since their actual evaluations are too laborious, we prefer to under mentioned approximate procedure.

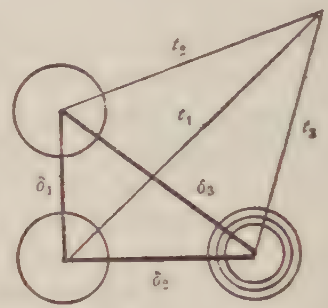

Fig. 3. Centers of two wave functions and a potential function.

Select one of the center, the point 1 in Fig. 3, say, as the origin of the coordinate system and expand the other function in terms of functions with the same origin. Then construct a sphere with the center at the origin whose radius being equal to the distance between the two centers. Let us assume that the potential whose center is situated in the third be approximated by

$$\begin{aligned} V(t_3) &= \text{const} = [V(\delta_2)+V(\delta_3)]/2 \qquad && t_1<\delta_1, \\ V(t_3) &= V(t_1)(\delta_1/\delta_3) && t_1>\delta_1. \end{aligned} \tag{4.4}$$

Since the three $\delta_i(i=1,\ 2,\ 3)$ values are nearly the same and most contributions to the integral are confined to the region where the overlapping of the electron could are relatively dense, this assumption may hold in a rather good approximation. If this approximation is legitimate, the evaluation of the three center integrals are reduced to the case of the two-centers.

The results of calculations for the case of III and IV mentioned in §2. are tabulated in the III and IV row of Table 2. The figures in the row indicated by E are the values due to Ivey's experimental formula (1.1). LiBr and LiI are not observed in experiment and they are extrapolated values according to this formula.

In Fig. 4 results of calculation of peak wave numbers for several lithium halides are plotted. Curves indicated by I' II, III, IV corresponds to respective potential function plotted in Fig. 1. We can see how the calculated curves approach to experimental values as the precision of our model increases. Fig. 5 is the plot of the two terms, the ground states and combining excited state mentioned above, as the function of interionic distance.

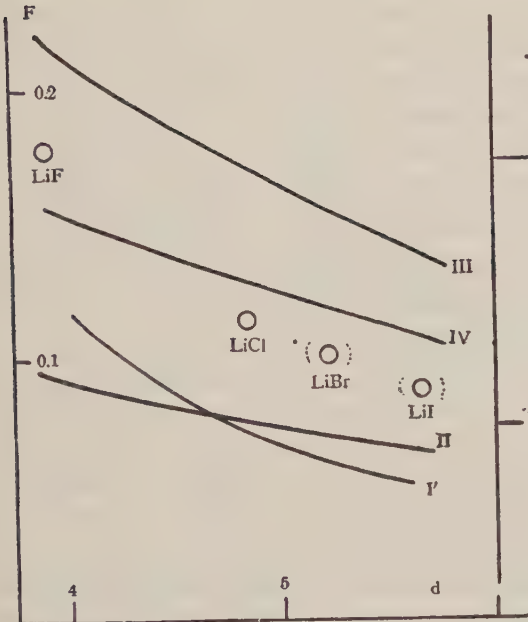

Fig. 4. Relation between peak of bands and inter-ionic distance for Li-salts (atomic unit).

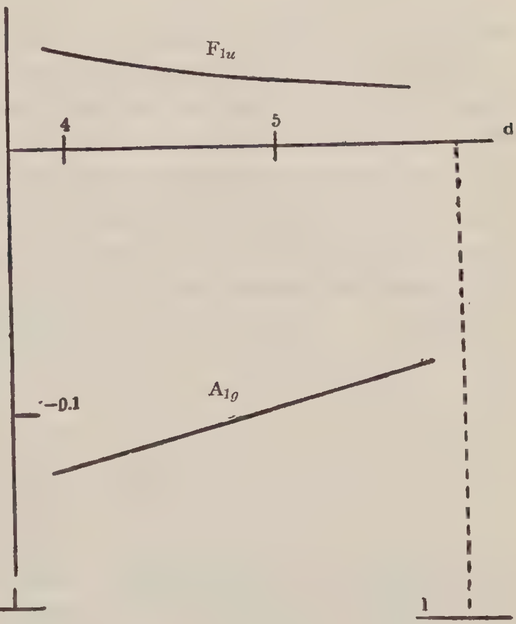

Fig. 5. Energy levels of trapped electron for Li-salts (atomic unit).

The numerical values appeared in the row I′ of Table 2 and the plot I′ in Fig. 4 correspond to the case where the polarization effect is also taken into account.

§5. Formulation of the dipole field.

The results obtained in the last paragraph have been deduced by the model of rigid lattice. Here we consider how to formulate the polarization field in the theory. We represent the field V by the sum of two terms.

$$V(\vec{r}) = V_R(\vec{r}) + V_P(\vec{r}) \tag{5.1}$$

The first term is the field due to the rigid lattice and the second is the field due to the dipoles which are induced by the polarization and deformation of ions owing to the lattice defect.

If we denote the dipole of i-th lattice point $\vec{p}_i$ and take the approximation like (2.2), V is expressed by the form:

$$V_P(\vec{r}) = \sum_{eff.} (\vec{r} - \vec{R}_i, \vec{p}^i) \mid \vec{r} - \vec{R}_i \mid^{-3} + \int_{all-eff.} (\vec{r} - \vec{R}, \vec{\mu} dz_R) \mid \vec{r} - \vec{R} \mid^{-3}. \tag{5.2}$$

Here the first sum is to be covered over the range involving ions which constitute the large molecule mentioned in § 2, and the remaining terms have been replaced by the integrals approximately assuming a effective continuous distribution of dipole moment.

If we assume that the density of the dipole moment vector $\vec{\mu}$ is directed toward the vacancy as it is represented in Fig. 6, in virtue of the symmetry consideration, and further assume that it is a function of R only, the integration of the second terms of (5.2) is easily carried out and get the following results.

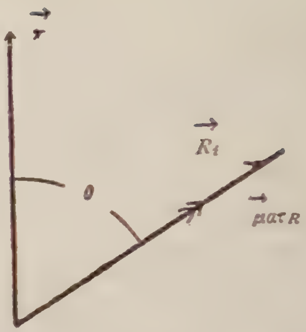

Fig. 6.
Direction of dipole density.

$$V(\vec{r})=\sum(\vec{r}-\vec{R}_i,\ \vec{p}_i)\ |\vec{r}-\vec{R}_i|^{-3}+\begin{cases}0 & |\vec{r}|<a\\ \int_r^\infty |4\pi\mu|\,dR & |\vec{r}|>a\end{cases} \tag{5.3}$$

where the parameter a means the radius of equivalent sphere substituting the portion of the large molecule in a continuum. Assuming that the intensity of the total field is directed along the radial direction in the domain $|\vec{r}|>a$, we can put the next relation.

$$|\vec{\mu}|=-\chi_{eff}\frac{\partial V}{\partial R}. \tag{5.4}$$

For simplicity, we consider only the case where $\chi_{eff.}$ is a constant. In this case, we obtain from (5.1), (5.3) and (5.4) the final result:

$$V(\vec{r})=\begin{cases}V_R(\vec{r})+\sum_{eff.}(\vec{r}-\vec{R}_i,\ \vec{p}_i)\ |\vec{r}-\vec{R}_i|^{-3} & |\vec{r}|<a\\ (x_{eff.}^{-1})[V_R(\vec{r})+\sum_{eff.}(\vec{r}-\vec{R}_i,\ \vec{p}_i)\ |\vec{r}-\vec{R}_i|^{-3}] & |\vec{r}|>a\end{cases} \tag{5.5}$$

$$x_{eff.}=1+4\pi\chi_{eff.} \tag{5.6}$$

where $x_{eff.}$ is the effective dielectric constant. The above mentioned effect of polarization is schematically represented in the primed figures of Fig. 1. Each primed figure is obtained from the corresponding unprimed one by adding the term $V_P(\vec{r})$.

Now, the problem how to determine the values of $\vec{p}_i$, and $x_{eff.}$ will be touched in the following. This is a fairly difficult problem because the mechanism of displacement and deformation of ions is very complicated. Several authors treated the matter from the stand point of lattice statistics.[13],[14] But in the case of F-center, both $\vec{p}_i$ and $x_{ff.}$ depend on the form of the wave function Ψ of the trapped electron. Therefore we must treat the matter in a self consistent manner, and a

detailed calculation of the polarization effect will be postponed to another opportunity.

Here, it may be interesting to compare the field discussed here to the one used by Tibbs.[(5)] Considered from the point of the above discussion, shown schematically in Fig. 1, the Tibbs's potential is near to our continuum model II′.

If we regard the results of § 4 as the zeroth approximation, the effect due to the term $V_P(\vec{r})$ may be considered as the perturbation. On the other hand, if we treat the effect of polarization from the outset, it is natural to start from the simplest medel I′. Then the effective dielectric constant $\varkappa_{eff.}$ is determined by

$$\varkappa_{eff.}=1+4\pi(\alpha_+ + \alpha_-)/(2d)^3$$

where α_+ and α_- denote the polarizability of the free positive and negative ions respectively. $\varkappa_{eff.}$ corresponds to $\varkappa_0$, the optical dielectric constant, which has been used by Tibbs. The values of $\varkappa_{eff.}$ are tabulated in Table 3. Using these values of $\varkappa_{eff.}$, we have calculated the peak wave number according to our model I′. They are represented in the row I′ of Table 2 and plotted in Fig. 4. It is remarked that the dependence of the wave number to the variation of the positive ions with fixed negative ion is not in accord with that of the experimental results.

	F	Cl	Br	I
Li	1.757	2.366	2.450	2.659
Na	1.621	2.087	2.175	2.357
K	1.617	1.911	1.995	2.132
Rb	1.683	1.909	2.025	2.089

Table 3. Effective dielectric constant.

As we have stated in the last paragraph, the calculated peak wave numbers approach systematically to the observed ones as our order of approximation increases in spite of our assumption of the rigid lattice model. Thus the authors are of opinion that their model fits rather good to the actual circumstance and the effect of polarization does not play an important part at least in the case of determination of the wave number of F-absorption bands.

§ 6. Summary.

(1) In order to obtain the value of a peak number of F-bands, we have studied systematically the potential field for the trapped electron in the vicinity of the vacancy. In the zeroth approximation, we employ the model of the rigid lattice neglecting the effects of polarization. The potential fields used in our theory are schematically shown in Fig. 1.

(2) For several cases of the potential fields, the Schroedinger equation was solved by the variation method. For the continuum model we took the form of the hydrogen-like wave function. As to the model of the large molecule, the form of the wave function is the linear combination of the six $2S$ wave functions of the Li-atom located at the nearest lattice points.

(3) The results of calculations are tabulated in Table 2, which show that the

calculated energy approaches to the observed ones as the precision of our model increases.

(4) The effect of polarization has been formulated in the theory by assuming that the field is due to the dipoles located at the lattice points. The results are shown schematically as the primed figures in Fig. 1. The calculation due to the field Case I′ suggests that the polarization contributes no primary effect.

The authors wish to thank Mr. Obi for his kind admission to use the result of the Hartree-function for Li-atom before the publication. We are also very grateful to Dr. Muto, the Professor of the Institute of Science and Technology of Tokyo University, for his valuable discussions and thanks are due to Miss Mikosiba for her assistance in the numerical work. The authors are indebted to the Japanese Council of Science for their financial help.

Mathematical Appendix

I) Solution of the secular equation

From (3.6) and (3.7) the secular equation is set as follows :

$$\begin{vmatrix} 0_A & 0_B & 0_B & 0_C & 0_B & 0_B \\ 0_B & 0_A & 0_B & 0_B & 0_C & 0_B \\ 0_B & 0_B & 0_A & 0_B & 0_B & 0_C \\ 0_C & 0_B & 0_B & 0_A & 0_B & 0_B \\ 0_B & 0_C & 0_B & 0_B & 0_A & 0_B \\ 0_B & 0_B & 0_C & 0_B & 0_B & 0_A \end{vmatrix} = 0, \tag{A.1}$$

where

$$\begin{aligned} 0_A &= \int \psi'_{100}(H-E)\psi'_{100}\, d\tau, \\ 0_B &= \int \psi_{100}(H-E)\psi_{100}\, d\tau, \\ 0_C &= \int \psi_{-100}(H-E)\psi_{100}\, d\tau. \end{aligned} \tag{A.2}$$

(A. 1) is reduced to the following three equations.

$$\begin{aligned} &0_A - 0_C = 0, \\ &0_A + 40_B + 0_C = 0, \\ &0_A - 20_B + 0_C = 0. \end{aligned} \tag{A.3}$$

The first two equations of (A. 3) give the energy values $E^{(2)}$ and $E^{(1)}$ as follows :

$$\begin{aligned} E^{(2)} &= I + (A-C)/(1-S_C), \\ E^{(1)} &= I + (A+B+C)/(1+4S_B+S_C), \\ I &= \int \psi_{100}(-\Delta/2 + V_{100})\psi_{100}\, d\tau. \end{aligned} \tag{A.4}$$

I means the ionization energy of the positive ion and A, B, C are calculated from the following formula according to the model of § 2.

Case III)

$$
\begin{aligned}
&z_1=1/6,\\
&A=-5A_{z11}+4A_{z12}+A_{z12},\\
&B=-4B_{z11}+2B_{z12}+2B_{z13},\\
&C=-4C_{z11}+4C_{z12}.
\end{aligned}
\tag{A. 5}
$$

Case IV $\quad z_1=1/2, \quad z_2=1/4, \quad z_3=1/8,$

$$A=(-A_{z11}+4A_{z12}+A_{z13})+(4A_{z21}+4A_{z22}+4A_{z23})+(4A_{z31}+4A_{z32}),$$

$$
\begin{aligned}
B=&(-2B_{z12}+2B_{z13})+(B_{z21}+4B_{z22}\\
&+2B_{z23}+4B_{z24}+B_{z25}+(2B_{z31}\\
&+4B_{z32}+2B_{z33}),
\end{aligned}
\tag{A. 6}
$$

$$C=4C_{z12}+(3C_{z21}+4C_{z22})+8C_{z31}.$$

All terms which appear in (A. 5) and (A. 6) are the matrix elements whose values are given by the integrals in which we put the product of (4.1), (4.2) or (4.3) as integrand. The position of center of these functions are shown in Table 4, (See Fig. 2) S_B and S_C in (A. 4) are the overlap integrals and are given as follows:

$$S_B=\int \psi_{010}\psi_{100}\,d\tau,$$

$$S_C=\int \psi_{-100}\psi_{100}\,d\tau.$$

II) Calculation of the two center integral (Coulson's method)

If we take the simplifying assumption (4.4), we can reduce the calculation of all matrix elements to the evaluation of two center integrals of form:

$$\int t_1^{m_1}\exp(-b_1t_1)\cdot t_2^{m_2}\exp(-b_2t_2)\,d\tau. \tag{A.8}$$

In order to transform the coordinate system denoted by the suffix 2 to that of 1 we can take the following formula which has been used in the Coulson's paper l. c.

Matrix-elements	Center of wave-function		Center of potential
S_B	(100)	(010)	
S_C	(100)	(−100)	
A_{z11}	(100)	(100)	(100)
A_{z12}	(100)	(100)	(010)
A_{z13}	(100)	(100)	(−100)
B_{z11}	(100)	(010)	(100)
B_{z12}	(100)	(010)	(001)
B_{z13}	(100)	(010)	(−101)
C_{z11}	(100)	(−100)	(100)
C_{z12}	(100)	(−100)	(010)
A_{z21}	(100)	(100)	(110)
A_{z22}	(100)	(100)	(011)
A_{z23}	(100)	(100)	(−110)
B_{z21}	(100)	(010)	(110)
B_{z22}	(100)	(010)	(101)
B_{z23}	(100)	(010)	(−110)
B_{z24}	(100)	(010)	(−101)
B_{z25}	(100)	(010)	(−1−10)
C_{z21}	(100)	(−100)	(110)
C_{z22}	(100)	(−100)	(011)
A_{z31}	(100)	(100)	(111)
A_{z32}	(100)	(100)	(−111)
B_{z33}	(100)	(010)	(111)
B_{z31}	(100)	(010)	(−111)
B_{z32}	(100)	(010)	(−1−11)
C_{z31}	(100)	(−100)	(111)

Table 4. Centers of potential function and wave function (See Fig. 2).

$$t_2^{-1} \exp\ (-b_2 t_2) = \sum_{n=0}^{\infty} (2n+1)\,(t_1\rho)^{-\frac{1}{2}} P_n\ (\mathrm{coh}\ \theta_b)\gamma_n(b_2 t_1\,;\rho),$$

$$\gamma_n = \begin{cases} I_{n+1/2}(b\rho)K_{n+1/2}(bt_1) & \rho > t_1 \\ I_{n+1/2}(bt_1)K_{n+1/2}(b\rho) & \rho < t_1 \end{cases} \qquad (A.\ 9)$$

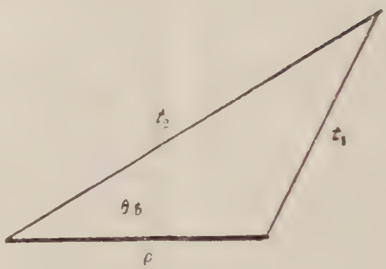

Fig. 7.

where I and K are the modified Bessel functions and θ_b and ρ are shown in Fig. 7.

Then we can estimate (A. 8) in terms of the following two integrals.

$$\int_0^R r^m I_n(r)\ \exp\ (-ar)dr = P_{mn}(a, R),$$
$$\int_R^{\infty} r^m K_n(r)\ \exp\ (-ar)dr = Q^*_{mn}(a, R). \qquad (A.\ 10)$$

In this way the matrix elements are calculated in terms of the following quantities.

$$\begin{aligned}
A_n &= (2\pi)^{1/2} P_{n+1/2\ \ n+1/2}\ [R], & F^* &= (2/\pi)^{1/2} Q^*_{n+1/2\ \ n+1/2}\ [R], \\
B_n &= (2\pi)^{1/2} P_{n+1/2\ \ n+1/2}, & G^* &= (2/\pi)^{1/2} Q^*_{n+1/2\ \ n+1/2}, \\
C_n &= (2\pi)^{1/2} P_{n+1/2\ \ n-1/2}, & H^* &= (2/\pi)^{1/2} Q^*_{n+1/2\ \ n-1/2}, \\
D_n &= (2\pi)^{1/2} P_{n+3/2\ \ n-1/2}, & M^* &= (2/\pi)^{1/2} Q^*_{n+3/2\ \ n-1/2}, \\
E_n &= (2\pi)^{1/2} P_{n+5/2\ \ n-1/2}, & N^* &= (2/\pi)^{1/2} Q^*_{n+5/2\ \ n-1/2}, \\
\tilde{E}_n &= (2\pi)^{1/2} P_{n+7/2\ \ n-1/2}, & \tilde{N}^* &= (2/\pi)^{1/2} Q^*_{n+7/2\ \ n-1/2}
\end{aligned} \qquad (A.\ 11)$$

where

$$P_{mn}[R] = R^m I_n(R)\ \exp\ (-aR),$$
$$Q^*_{mn}[R] = R^m K_n(R)\ \exp\ (-aR).$$

These quantities are estimated by the following recurrence formula due to Coulson.[12]

$$\begin{aligned}
&A_0 = (e^R - e^{-R})e^{-aR}, \qquad A_{-1} = (e^R + e^{-R})e^{-aR}/R, \\
&A_n = R^2 A_{n-2} - (2n-1)A_{n-1}, \\
&B_0 = (a-1)^{-1}(1 - e^{-(a-1)R}) - (a+1)^{-1}(1 - e^{-(a+1)R}), \\
&(a^2 - 1)B_n = 2nB_{n+1} - aA_n - RA_{n-1}, \\
&C_n = aB + A_n, \qquad D_n = B_{n+1} + (2n+1)B_n, \\
&E_n = C_{n+2} + (2n+1)C_{n+1}, \quad \tilde{E}_n = D_{n+2} + (2n+1)D_{n+1}, \\
&F^*_0 = e^{-(a+1)R}, \qquad F^*_{-1} = e^{-(a+1)R}/R, \\
&F^* = R^2 F^*_{n-2} + (2n-1)F^*_{n-1},
\end{aligned} \qquad (A.\ 12)$$

$$G_0^* = e^{-(a+1)R}/(a+1),$$
$$(a^2-1)G_n^* = -2nG_{n-1}^* + aF_n^* - RF_{n-1}^*, \tag{A. 13}$$
$$H_n^* = -aG_n^* + F_n^*, \qquad M_n^* = G_{n-1}^* - (2n+1)G_n^*,$$
$$N_n^* = H_{n+2}^* - (2n+1)H_{n+1}^*, \quad \tilde{N}^* = M_{n+2}^* - (2n+1)M_{n+1}^*.$$

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On the Relativistically Improved Integration in Perturbation Theory.*

—Its Meaning in Feynman Theory—

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§ 1. Introduction.

Previously,[1] we proposed a new method to integrate in relativistically invariant manner the integral appearing in perturbation calculation. Hitherto, in various problems of the quantum field theory, the customary perturbation method has frequently given results evidently destroying the relativistic covariance; for example, in the problem of the self-energy of a moving electron. But, since the system of quantum field theory is of a relativistically invariant structure, it must be concluded that the cause of the failure of relativistic covariance lies in the course of the perturbation calculation. Our new calculation method (w-method) remedies this defect of perturbation calculation and gives results having the correct relativistic covariance.

The advantage of the w-method are as follows; results having the correct relativistic covariance can be obtained; the calculation becomes much easier and the relation to the various results obtained by means of usual perturbation calculation can be seen explicitly.

However, we apply this method to the usual perturbation formalism which has not the covariant form. Of course, such patchy method will soon see its ruin as his own natural fate. It comes with the appearance of Feynman's theory![2] But available parts of the past theory always still remain, and moreover, the perturbation calculation in the quantized field theory is a relativistic invariant method in spite of its non-relativistic form. In the calculation in Feynman theory, w-method can also be used and its meaning becomes clear. In the w-method, the domain of integral is determined in relativistically invariant manner. Considering the physical meaning of this method determining the domain, w-method is easily introduced into Feynman theory.

§ 2. w-method in Feynman theory.

For simplicity, we consider the case when particles with p_μ, k_μ (mass $\sqrt{\Delta_1}$, $\sqrt{\Delta_2}$ respectively) are created, annihilating a particle with energy momentum 4-vecter t_μ (mass x).

* Continued from Prog. Theor. Phys. 4 (1949), 420.

Let us discuss the integral in following most simple case.

$$I(\Delta_1,\ \Delta_2)=\int\frac{dpdk\delta(p+k-t)}{(p^2-\Delta_1)\ (k^2-\Delta_2)} \tag{1}$$

$$=\int\frac{dk}{\{(k^2-2(tk)+\varkappa^2-\Delta_1\}\,(k^2-\Delta_2)} \tag{1'}$$

In order to investigate this integral on the bases of the physical meaning of w-method, we shall first investigate what domain should be taken in the integration by w-method.

i) w-method point of view.

In the perturbation calculation, an integral of the following form occurs in the present case;

$$\int\int F\,(p,\ k)d\boldsymbol{p}\,d\boldsymbol{k} \tag{2}$$

where $F(p,\ k)$ is a certain function of p_μ and k_μ, and bold letters mean three dimensional space vectors. Owing to the momentum conservation law (1) can be written as

$$\int G(\boldsymbol{p})d\boldsymbol{p}. \tag{3}$$

In the w-method, the domain of integral is determined as follows:

(a) The integration domain must be prescribed for the integral (2).

(b) The domain is a region enclosed by a surface on which the momentum-space scalar quantity w takes a constant value.

(c) The domain becomes a momentum space sphere when referred to a appropriate system of coordinates.

(d) As the four dimensional scalar quantity w, we may take the four dimensional scalar product of the momentum energy four vectors (p, k). Then we transform the integral variable p to $w(p, k)$ and integrate in domain between two surfaces w=const.

Accordingly this integration domain is completely symmetric for two particles p, k, in the intermediate state, and the contributions of two particles with mass $\sqrt{\Delta_1}$, $\sqrt{\Delta_2}$ to the integral (2) (Fig. 1, C_1 and C_2) are unified to the domain of w on an average.

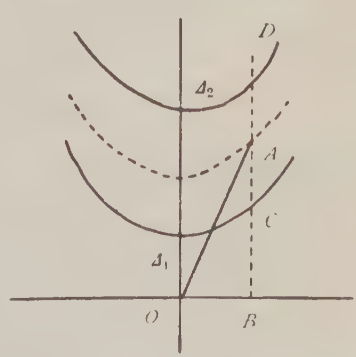

Fig. 1.

In our previous calculation, however, we used three-dimensional angle $\cos\theta$ and only after integrating for θ, we obtained the only w-dependent function without involving t_μ i. e. relativistically invariant form, since (w, θ) does not construct an orthogonal coordinate system. That the result becomes to have the only w-dependent form, after integrating for θ shows that w is the inner product of

energy-momentum four vectors belonging to two particles with oppositely directed momenta $\boldsymbol{l}$ and $-\boldsymbol{l}$ respectively, because, by the condition (c), the domain of w is a sphere when referred to the coordinate system $\boldsymbol{p}+\boldsymbol{k}=0$, $\boldsymbol{t}=0$.

$$w=E_{\Delta_1}E_{\Delta_2}+(\boldsymbol{l},\boldsymbol{l}),\tag{4}$$

$$E_{\Delta}=(l^2+\Delta)^{\frac{1}{2}}$$

and so, from (4) (cf. Fig. 1. curve w)

$$w=\overline{OA}^2=\overline{OB}^2+\overline{BA}^2,\qquad \overline{BA}^2=\overline{BC}\cdot\overline{BD}.\tag{5}$$

ii) w-method in Feynman theory.

On the basis of the above consideration, let us consider the integration method of (1′). One of the differences between the interpretation of perturbation theory and that of Feynman theory is that in the former all physical phenomena are regarded as the motions subjected to a definite time direction while in the latter such transitions as the pair-production as motions in opposite time direction. Taking into account the motion in opposite time direction as Feynman, we can rewrite the integral $\int_{-\infty}^{\infty}dt_n\cdots\int_{-\infty}^{t_3}dt_2\int_{-\infty}^{t_2}dt_1$ in perturbation theory into $\frac{1}{n!}\int_{-\infty}^{\infty}dt_n\cdots\int_{-\infty}^{\infty}dt_2\int_{-\infty}^{\infty}dt_1$, because we have then no longer the restrictions $t_1<t_2$ etc. This situation makes following difference between two theories. In perturbation theory, the integration is over the freedom of energy, since there it is unnecessary to conserve energy.

On the other hand in Feynman theory, energy and momentum are conserved, so that there is no freedom of energy for a definite mass. Because of the relation $p+k=t$ in (1), however, it is not always necessary that mass in $p+k$ is Δ_1; and so, there is the freedom of mass! Accordingly, in transforming this freedom of mass to the freedom of energy for definite masses Δ_1 and Δ_2, one can make obvious the relation between perturbation and Feynman theory.

In order to introduce the freedom of mass explicitly, we take the following procedure. According to the consideration in (i), we symmetrize the integral (1′) for p and k. In performing this, we use a brilliant formula introduced by Feynman.[2] (The essential meaning of this formula is this !!!)

$$a^{-1}b^{-1}=\int_0^1dx(ax+b(1-x))^{-2}.\tag{6}$$

Using this, (1′) becomes)

$$I(\Delta_1,\Delta_2)=\int_0^1dx\int\frac{d^4k}{\{k^2-2(t,k)x+(\Delta_2-\Delta_1+x^2)x-\Delta_2\}^2}$$

$$=\int_0^1dx\int\frac{d^4l}{\{l^2-(x^2x^2-Z)\}^2},\tag{7}$$

$$l=k-xt,\tag{8}$$

$$Z=(\varDelta_2-\varDelta_1+x^2)x-\varDelta_2. \tag{9}$$

Thus, we can write down the effect of two particles k, p by only one variable l; and then, we take four dimensional sphere of l as integration domain. (This corresponds to the specification of intergation domain of w!)

Consequently, we can interpret the integral (7) as the contributions from particles with the various masses $m(m^2=x^2x^2-Z)$. (Of course, this means only that there is the freedom of mass, but not that various particles with various masses really exist.) In order to complete the correspondence to w, it is necessary that contributions from these particles with various masses should be reduced to that from two particles with masses $\varDelta_1$ and $\varDelta_2$. The domain $x=[1,0]$ corresponds to the domain $m^2=[\varDelta_1, \varDelta_2]$, that is, (7) shows in Fig. 1 that mass can vary only in the domain between curve C_1 and C_2. Performing the integral of (7) for x, these contributions of particles with various masses collectively become those of particles with mass $\varDelta_1$ and $\varDelta_2$ i. e. the values of both end in the domain (cf. Fig. 1. cuves C_1 and C_2).

Carrying out the integrations in l_4 and x,

$$I(\varDelta_1, \varDelta_2)=\frac{\pi^2 i}{(2\pi)^2 2x^2}\int_0^\infty \frac{l^2 dl}{\left\{l^2-\frac{1}{4x^2}(\varDelta_1+\varDelta_2-x^2)^2+\frac{\varDelta_1\varDelta_2}{x^2}\right\}}$$

$$\times\left[\frac{x^2-(\varDelta_2-\varDelta_1)}{E_{\Delta_1}}+\frac{x^2-(\varDelta_1-\varDelta_2)}{E_{\Delta_2}}\right], \quad l=|\boldsymbol{l}|. \tag{10}$$

Of course, (10) is of a perfectly symmetric form for $\varDelta_1$ and $\varDelta_2$, and shows contributions from two curves C_1 and C_2 in Fig. 1.

Now, the correspondence to w is complete.

w is related to l by (4). However, l is not a relativistically invariant quantity but w an invariant quantity. Therefore, transforming the variable l to w by (4), from (10),

$$I(\varDelta_1, \varDelta_2)=\frac{2\pi^2 i}{(2\pi)^2}\int_{\sqrt{\Delta_1\Delta_2}}^{w\infty} dw\,\frac{\left(\frac{w^2}{4}-\varDelta_1\varDelta_2\right)^{1/2}[x^2(w+\varDelta_1+\varDelta_2)-(\varDelta_1-\varDelta_2)^2]}{(x^2w^2-fw+g)(w+\varDelta_1+\varDelta_2)} \tag{11}$$

$$\left.\begin{aligned} f&=(\varDelta_1-\varDelta_2)^2-2x^2(\varDelta_1+\varDelta_2)+x^4\\ g&=2x^2(\varDelta_1^2+\varDelta_2^2)-x^4(\varDelta_1+\varDelta_2)-(\varDelta_1-\varDelta_2)^2(\varDelta_1+\varDelta_2)\end{aligned}\right\} \tag{12}$$

Thus, we can express the integral (1) by using an invariant in energy momentum space. Since it is an expression in energy-momentum spacc, its relation to previous results produced by means of the ordinary perturbation calculation is clear. By using this procedure, as we shall show later, we can obtain the same results as in the usual perturbation calculation.

Summary: We may rewrite integration in general case, by using formula (6) repeatedly and by transformation (8), to the form of integral (7) for the

quantity l which is symmetric for two particles p, k, ; and then, transform this quantity l to w by using (4).

At that time, the order of integration for x and l_4, the transformation $l\to w$ and integration for l is arbitrary. After all, one must transform variable l to w somewhere, and bring the expression to the Lorentz invariant form.

§ 3. Example, self-energy of an electron.

For example, we shall calculate the self-energy of an electron, and show its results to agree with the previous results by using w-method in perturbation theory.[2]

Self-energy of an electron ψ (mase $\sqrt{\Delta_1}=\varkappa$) due to the field ϕ (mass $\sqrt{\Delta_2}$) is, taking the interaction $f\bar{\psi}O\psi\phi$ (O: a function of γ-matrices), given by

$$W=+\bar{\psi}(t)\frac{if^2}{\pi}\int\frac{d^4pd^4kO[(\gamma p)+\varkappa)]O}{(p^2-\varkappa^2)(k^2-\Delta_2)}\delta(p+k-t)\psi(t) \tag{13}$$

(i) Self-energy of an electron due to the neutral scalar meson: $W_f(O=1)$ (e. g. Self-energy of an electron due to the C-meson)
In this case, we have following two integrals:

$$I(\Delta_1,\Delta_2)=\int\frac{d^4pd^4k\delta(p+k-t)}{(p^2-\Delta_1)(k^2-\Delta_2)} \tag{3}$$

$$I_\sigma(\Delta_1,\Delta_2)=\int\frac{k_\sigma d^4pd^4k\delta(p+k-t)}{(p^2-\Delta_1)(k^2-\Delta_2)} \tag{14}$$

Using the above mentioned procedure,

$$I(\Delta_1,\Delta_2)=+\frac{i}{2}\Big[\frac{1}{2}\log\frac{2w}{\varkappa^2}-\frac{1}{4\varkappa^2}\Big\{(\varkappa^2+\Delta_1-\Delta_2)\ \log\Big(\frac{\Delta_1}{\varkappa^2}\Big)$$

$$+(\varkappa^2-\Delta_1+\Delta_2)\ \log\Big(\frac{\Delta_2}{\varkappa^2}\Big)\Big\}-\frac{2D}{4\varkappa^2}\ \log\Big\{\frac{1}{2}\Big(\frac{\Delta_1}{\Delta_2}\Big)^{1/2}\frac{(\Delta_2+\Delta_1-\varkappa^2-D)}{\Delta_2}\Big\}\Big]_{w\to\infty} \tag{11'}$$

$$I_\sigma(\Delta_1,\Delta_2)=+\frac{i}{4}t_\sigma\Big[\frac{1}{2}\ \log\frac{2w}{\varkappa^2}-\frac{1}{2\varkappa^2}(\Delta_1-\Delta_2)-\frac{1}{2\varkappa^2}\Big\{\Delta_1\ \log\Big(\frac{\Delta_1}{\varkappa^2}\Big)$$

$$-\Delta_2\log\Big(\frac{\Delta_2}{\varkappa^2}\Big)\Big\}-\frac{B}{4\varkappa^4}\Big\{(\varkappa^2+\Delta_1-\Delta_2)\ \log\Big(\frac{\Delta_1}{\varkappa^2}\Big)+(\varkappa^2-\Delta_1+\Delta_2)\ \log\Big(\frac{\Delta_2}{\varkappa^2}\Big)\Big\}$$

$$+\frac{2BD}{4\varkappa^4}\log\Big\{\frac{1}{2}\Big(\frac{\Delta_1}{\Delta_2}\Big)^{1/2}\frac{(\Delta_2+\Delta_1-\varkappa^2-D)}{\Delta_2}\Big\}\Big]_{w\to\infty}. \tag{14'}$$

$$B=\varkappa^2-\Delta_1+\Delta_2,\quad D=(B^2-4\varkappa^2\Delta_2)^{1/2}.$$

Substituting the masses of electron and C-meson, Δ_1, Δ_2 ($\Delta_1=\varkappa^2$, $\Delta_2=m^2$) into (11′) and (14′), we can obtain the same results as before.[3] But we can get previous results even in the form of integral. Performing the transformation in (11),

$$w'=2w+\Delta_2, \tag{15}$$

$$I(x^2,\Delta_2)=\frac{\pi^2 i}{(2\pi)^2}\int_{m^2+2\kappa m}^{\infty}\frac{\{(w'+\Delta_2)^2-4\Delta_2(w'+\Delta_1)\}^{1/2}}{w'(w'+\Delta_1)}dw'. \tag{16}$$

Similarly,

$$I_o(x^2,\Delta_2)=\frac{\pi^2 i}{2(2\pi)^2}t_o\left\{\int_{m^2+2\kappa m}^{\infty}\frac{\{(w'+\Delta_2)^2-4\Delta_2(w'+\Delta_1)\}^{1/2}}{w'(w'+\Delta_1)}dw'\right.$$
$$\left.-(\Delta_1-\Delta_2)\int_{m^2+2\kappa m}^{\infty}\frac{\{(w'+\Delta_2)^2-4\Delta_2(w'+\Delta_1)\}^{1/2}}{w'(w'+\Delta_1)^2}dw'\right\}, \tag{17}$$

From (13), $(O=1)$

$$W'_f=+\bar\psi(t)\frac{if^2}{\pi}\Big\{((\gamma t)+x)I-(\gamma_o I_o)\Big\}\psi(t)$$
$$=-\frac{f^2}{2\pi}x\left[\frac{2}{4}\int_{m^2+2\kappa m}^{\infty}\frac{\{(w'+\Delta_2)^2-4\Delta_2(w'+\Delta_1)\}^{1/2}}{w'(w'+\Delta_1)}dw'\right\}.$$
$$+\frac{(\Delta_1-\Delta_2)}{4}\int_{m^2+2\kappa m}^{\infty}\frac{\{(w'+\Delta_2)^2-4\Delta_2(w'+\Delta_1)\}^{1/2}}{w'(w'+\Delta_1)^2}dw'\bigg]\bar\psi(t)\psi(t). \tag{18}$$

This perfectly agrees with the previous result obtained by means of w-method in usual perturbation theory.[3]

ii) Electromagnatic self-energy of an electron $W_e(\Delta_2=0, f=e, O=i\gamma_\mu, \phi=A_\mu)$. Using (13)

$$W_e=\frac{ie^2}{\pi}\bar\psi(t)\{\gamma_\mu(\gamma,t+x)\gamma_\mu I-\gamma_\mu\gamma_o I_o\gamma_\mu\}\psi(t)$$
$$=-\frac{ie^2}{\pi}\bar\psi(t)\left\{2I+2\frac{\partial I_o}{\partial t_o}\right\}x\psi(t). \tag{19}$$

Setting $\Delta_1=x^2$, and $\Delta_2=0$ in (11′) and (14′) (accordingly $B=D=0$)

$$I(x^2,0)=\frac{i}{2}\frac{1}{2}\log\left(\frac{2w}{x^2}\right)_{w\to\infty}, \tag{20}$$

$$I_o(x^2,0)=\frac{i}{4}t_o\left\{\frac{1}{2}\log\left(\frac{2w}{x^2}\right)-\frac{1}{2}\right\}_{w\to\infty}. \tag{21}$$

From (19), (20), (21)

$$W_e=\frac{3e^2}{2\pi}x\left\{\frac{1}{2}\log\left(\frac{2w}{x^2}\right)-\frac{1}{6}\right\}\psi(t)\psi(t). \tag{22}$$

This also agrees with the previous result.[3]

In conclusion, we would like to express our sincere thank to Prof. S. Sakata, for his interest in this work and to Mr. Minoru Umezawa for his valuable discussions.

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Problems of Ambiguity in Quantum Field Theory

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§ 1. Introduction.

The covariant quantum field theory developed by Tomonaga, Schwinger and Feynman has been applied by many authors to the enormous caluculations of various processes. However, difficulties which seemed to attributable to the mathematical defects in usual quantum electrodynamics still appear with the various types everywhere in these calculations. Although these causes may be analyzed from the different points of view, one can treat some of them as the problems of ambiguity by distinguishing them from the fatal difficulties of so-called divergence.

A part of these problems is characterized by the aspect that the results of calculations of matrix elements contradict with the formal requirements, such as the gauge-invariance, divergence theorem and equivalence theorem. Although these discrepancies would be removed by formal procedure, there would still remain various ambiguities in the results which could not be tested by any requirement, so that these conclusions are quite unreliable.

Recently, Pauli and Villars[1] proposed the regulator method by which one can automatically remove these difficulties, and the circumstances seemed to be much improved. However, this method has not only the own difficulties that its procedure contradicted with the present concepts of field theory, but also has no definite rule on what conditions one may rationalized the results. Accordingly, we can hardly find the consistent conditions by which one separate physically significant results from non-physical terms throughout in the field theory.[2]

On the other hand, the method of the mixed field theory analyzed by Umezawa and Kawabe, Feldman and Rayski[3] succeeded in the problem of photon self-energy, but not offered the answers for the removal of ambiguity in general forms.

In this paper, we try to examine whether the method satisfying the following three requirements simultaneously exists or not:

1. The matrix elements which have the formal properties, such as the gauge invariancy, divergence theorem and equivalence theorem should also preserve them after the calculations.
2. The ambiguous terms in the matrix elements which have no formally required propeities should also be removed.

3. All physically significant terms should be retained after the removal of ambiguous terms.

It will be concluded that we can hardly find the mathematically consistent theory satisfying above these requirements. In spite of this conclusion, if we utilize the points of the mathematical defects in the opposite sense, we may find a way of the temporal escape, though it seems very paradoxical.

A procedure will be proposed in this paper as an example of such a method. From the results obtained in this way, we will conclude that the regulator method has been raised the unnecessary confusions and more analytical method should be necessitated by which one can overcome these difficulties throughout.

§ 2. General Remarks and the Photon Self-Energy Problem.

Firstly, we analyze what happens from the pathological nature of the invariant delta-functions in the calculations. To our opinion, the causes by which the ambiguous terms are raised, are partly due to the fact that in spite of the circumstances that these pseudo-functions are not well defined at the origin, we are obliged to evaluate the values of them at very this point. Consequently, in order to obtain any definite result, we should have recourse to the so-called "Arithmetics of Infinity", such as $\infty \pm \infty$, $\infty \times \infty$, $\frac{\infty}{\infty}$ etc..

Accordingly, although these functions would satisfy the identity

$$(\Box^2 - m^2)\Delta(X) = 0 \quad \text{and} \quad (\Box^2 - m^2)\bar{\Delta}(X) = -\delta(X), \tag{2.1}$$

we can not assert that this very nature is preserved in the calculations of the matrix elements which consist of these functions. For instances, we have the following identities from the above mentioned nature:

$$\int (q_\lambda^2 + L)\,\delta^{(n)}(q_\lambda^2 + L)\,dq = -n\int \delta^{(n-1)}\ (q_\lambda^2 + L)\,dq, \tag{2.2}$$

and more generally,

$$\int (q_\lambda^2 + L)^{n+\alpha}\delta^{(n)}(q_\lambda^2 + L)\,dq = 0 \quad \alpha \geq 1$$
$$\int (q_\lambda^2 + L)^{n-\alpha}\delta^{(n)}(q_\lambda^2 + L)\,dq = (-1)^{n-\alpha}\,\frac{n!}{\alpha!}\int \delta^{(\alpha)}(q_\lambda^2 + L)\,dq \quad \alpha \geq 0. \tag{2.3}$$

However, if we apply the other method of calculation to the left hand side of (2.2), we have for the cases of $n \leq 1$

$$\int (q_\lambda^2 + L)\,\delta^{(n)}(q_\lambda^2 + L)\,dq = \frac{1}{2\pi}\int da\,(ia)^n e^{iaL}\int (q_\lambda^2 + L)\,e^{iaq\lambda^2}dq.$$

Here, using the integral values of

$$\int e^{iaq\lambda^2}dq = i\pi^2\frac{\varepsilon(a)}{a^2},$$
$$\int q_\mu q_\nu e^{iaq\lambda^2}dq = -\delta_{\mu\nu}\pi^2\frac{\varepsilon(a)}{2a^2}\left[\frac{1}{a} - c_1\varepsilon(a)\delta(a) - i\pi c_2\delta(a)\right], \tag{2.4}$$

we have

$$(i)^n \frac{\pi}{2} \int da e^{iaL}[-2a^{n-3}+2c_1 a^{n-2}\varepsilon(a)\delta(a)+2i\pi c_2 a^{n-2}\delta(a)+iLa^{n-2}]\varepsilon(a)$$

$$= -(i)^n \frac{n\pi}{2}\int da\varepsilon(a)a^{n-3}e^{iaL}-(i)^n\pi(1-c_1)P(a^{n-2}e^{iaL})_{a=0}$$

$$+(i)^{n+1}\pi^2 c_2\int da\varepsilon(a)a^{n-2}\delta(a)e^{iaL},$$

where we retain the indefinite terms in (2.4) with arbitrary constants c_1 and c_2, and take the principal value in the last second term. On the other hand, the right-hand side of (2.2) becomes

$$-(i)^n\frac{n\pi}{2}\int da\varepsilon(a)^{n-3}a^{iaL}.$$

Judging from above circumstances, in order to identify the above two results at least in the cases of $n\leq 1$, we have to retain the indefinite terms and take the arbitrary constants as

$$c_1=1 \qquad \text{and} \qquad c_2=0. \tag{2.5}$$

And using these conditions, we can also get the identities in the cases of $n\geq 2$.

On the contrary, there may exist the definitions of (2.4) by which one can take in the case of $n\leq 1$

$$\int\left[\frac{2-n}{2}q_\lambda^2+L\right]\delta^{(n)}(q_\lambda^2+L)dq=0. \tag{2.6}$$

In fact, we calculate the left-hand side of (2.6) and get

$$(i)^n\pi\left(\frac{2-n}{2}c_1-1\right)P(a^{n-2}e^{iaL})_{a=0}+(i)^{n+1}\pi^2 c_2\int da\varepsilon(a)a^{n-2}\delta(a)e^{iaL}.$$

Accordingly, if we take the conditions

$$c_1=\frac{2}{2-n} \qquad \text{and} \qquad c_2=0, \tag{2.7}$$

the above identities hold.

Comparing the above two types of conditions, we can conclude that the definitions (2.5) and (2.7) are consistent in the case of $n=0$, but not in the case of $n=1$ and if we take the one identity, we are obliged to give up the others.

By the way, it should be remarked that in the cases of $n>2$, (2.6) is compatible with (2.2) in virture of the relations between the definite integrals:

$$\int L\delta^{(n)}(q_\lambda^2+L)dq=(2-n)\int\delta^{(n-1)}(q_\lambda^2+L)dq \quad n>2 \tag{2.8}$$

From the above considerations, it seems to be probably true that in the case of $n=1$, (2.6) also holds and if this would be true, it will be doubtful why (2.6) does not hold in the case of $n=2$, that is:

$$\int L\delta''(q_\lambda^2+L)\,dq \equiv 0. \tag{2.9}$$

But in this case, the circumstances are not so good as others. Because the left-hand side of (2.9) has the definite value π and not zero. In spite of this, this very equation plays the important roles in the problems of ambiguity as discussed in the later section.

Here, we consider the problem of photon self-energy. Using the new method of Schwinger's III,[4] we have

$$\begin{aligned} G_{\mu\nu}(X) &= \frac{1}{2(2\pi)^7}\int dk\,dk'\,e^{i(k+k')X}[k_\mu k'_\nu + k'_\mu k_\nu - \delta_{\mu\nu}(kk'-m^2)]\left[\frac{\delta(k^2+m^2)}{k'^2+m^2}+\frac{\delta(k'^2+m^2)}{k^2+m^2}\right] \\ &= -\frac{1}{2(2\pi)^7}\int dp\,e^{ipX}(p_\mu p_\nu - \delta_{\mu\nu}p_\lambda^2)\int_0^1 du\,2u(1-u)\int dq\,\delta'(q_\lambda^2+u(1-u)p_\lambda^2+m^2) \\ &\quad + \frac{\delta_{\mu\nu}}{4(2\pi)^7}\int dp\,e^{ipX}\int_0^1 du\int dq[q_\lambda^2+2\{u(1-u)p_\lambda^2+m^2\}]\delta'(q_\lambda^2+u(1-u)p_\lambda^2+m^2). \end{aligned} \tag{2.10}$$

The first term of this result expresses the charge alteration and the effect of vacuum polarization, the second the photon self-energy which forms (2.6)-type. If we use (2.5), we have

$$\begin{aligned} \frac{1}{4(2\pi)^7}\int dp\,e^{ipX}\int_0^1 du\int dq[\{u(1-u)p_\lambda^2+m^2\}\delta'(q_\lambda^2+u(1-u)p_\lambda^2+m^2) \\ -\delta(q_\lambda^2+u(1-u)p_\lambda^2+m^2)], \end{aligned} \tag{2.11}$$

and this has a quadratic divergence or a finite value which are already calculated by many authors.*

The new result of Schwinger's calculations seems to be equivalent to our definition (2.6) from the above consideration. In fact, we have after some calculations

$$\begin{aligned} &\int dk\,dk'\,e^{i(k+k')X}\left[(k'^2+m^2)\frac{(k+k',k)}{(k+k')^2}+(k^2+m^2)\frac{(k+k',k')}{(k+k')^2}\right]\left[\frac{\delta(k^2+m^2)}{k'^2+m^2}+\frac{\delta(k'^2+m^2)}{k^2+m^2}\right] \\ &= -\frac{1}{2}\int dp\,e^{ipX}\int_0^1 du\int dq[2[u(1-u)p_\lambda^2+m^2\}+q_\lambda^2]\delta'(q_\lambda^2+u(1-u)p_\lambda^2+m^2), \end{aligned}$$

which indicates the equivalency of Schwinger's procedure and our's. And both procedures give the vanishing photon self-energy. However, as discussed above, the discrepancy between (2.7) and (2.5) arises, and we must overcome this difficulty by any other method in the future theory. Although the regulator method and the mixed theory also remove this discrepancy, we must criticize the

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* If we cut off the upper limit in the momentum space, finite terms will include the p_λ^2-term which is reduced to charge alteration type. But this seems to be incorrect from the covariant view point and it is hoped that the calculations are taken such as the result is independent of p_λ^2-term as shown by Wentzel[5] and Pauli-Villars.[1]

method from the fact what degrees it can grasp the difficulties in general and to our oppinion these methods seem to fail again.

After all, it seems to be probably true, except the above example, that the validity of (2.6) in the case of $n=1$ exists and then we try to examine what results we can get from the applications of (2.6) to the removal of ambiguous terms appearing in the various elemental processes.

By the way we also consider the validity of (2.6) in the case of $n=2$ in the last section.

§ 3. Removal of Ambiguous Terms. Case 1 A→B($A^{\pm}\to B^{\pm}+C$ and A→A)

In order to examine the ambiguity, we have to investigate the processes of which the virtual fermion field relates, such as the closed loop type in the language of Dyson-Feynman's diagram.

Af first, we consider the second order processes of the self-energies of boson due to the fermion field and the decay processes of boson into fermion passing through the virtual fermion field, such as $\pi-\beta$ decay.

Excluding the isotopic spin, the matrix elements are written

$$\langle[\psi^{+}\gamma_{A}\psi]\rangle_{0}=-g_{B}\int dX'U_{B}(X')K_{AB}(XX'),$$

$$K_{AB}(X)=S_{p}(\sum S^{(1)}(-X)\gamma_{A}\bar{S}(X)\gamma_{B})$$

$$=\frac{4}{(2\pi)^{7}}\int dk\,dk'\,e^{i(k+k')X}K_{AB}(k,k')\left[\frac{\delta(k^{2}+m^{2})}{k'^{2}+m^{2}}+\frac{\delta(k'^{2}+m^{2})}{k^{2}+m^{2}}\right], \tag{3.2}$$

$$K_{AB}(k,k')=\frac{1}{4}S_{p}[(-i\gamma k-m)\gamma_{A}(i\gamma k'-m)\gamma_{B}]. \tag{3.3}$$

Writing $K_{AB}(k,k')$ schematically such as [A-coupling, B-coupling], we have the following results, where we take γ_A and γ_B as 1 (scalar), γ_μ (vector), $\gamma_\mu\gamma_\nu$ (tensor), γ_5 (pseudoscalar), $\gamma_5\gamma_\mu$ (pseudovector) and $\gamma_5\gamma_\mu\gamma_\nu$ (pseudotensor) respectively:

$$\begin{aligned}
&[s,s] &&=kk'+m^{2},\\
&[s,v]_{\mu}=[v,s]_{\mu} &&=im(k-k')_{\mu},\\
&[s,t]_{\rho\sigma}=-[t,s]_{\rho\sigma} &&=-(k_{\rho}k'_{\sigma}-k_{\sigma}k'_{\rho}),\\
&[s,pt]_{\rho\sigma}=-[pt,s]_{\rho\sigma} &&=(k_{\alpha}k'_{\beta})\tfrac{1}{4}S_{p}[\gamma_{5}\gamma_{\rho}\gamma_{\sigma}\gamma_{\alpha}\gamma_{\beta}],\\
&[v,v]_{\mu,\nu} &&=k_{\mu}k'_{\nu}+k_{\nu}k'_{\mu}-\delta_{\mu\nu}(kk'-m^{2}),\\
&[v,t]_{\mu,\rho\sigma}=-[t,v]_{\rho\sigma,\mu} &&=-im[(k+k')_{\rho}\delta_{\mu\sigma}-(k+k')_{\sigma}\delta_{\mu\rho}],\\
&[v,pv]_{\mu,\nu}=-[pv,v]_{\nu,\mu} &&=(k_{\alpha}k'_{\beta})\tfrac{1}{4}S_{p}[\gamma_{5}\gamma_{\mu}\gamma_{\nu}\gamma_{\alpha}\gamma_{\beta}],\\
&[v,pt]_{\mu,\rho\sigma}=-[pt,v]_{\rho\sigma,\mu} &&=-im(k+k')_{\alpha}\tfrac{1}{4}S_{p}[\gamma_{5}\gamma_{\rho}\gamma_{\sigma}\gamma_{\mu}\gamma_{\alpha}],\\
&[t,t]_{\mu\nu,\rho\sigma} &&=(k_{\alpha}k'_{\beta})\tfrac{1}{4}S_{p}[\gamma_{\alpha}\gamma_{\mu}\gamma_{\nu}\gamma_{\beta}\gamma_{\rho}\gamma_{\sigma}]+m^{2}\tfrac{1}{4}S_{p}[\gamma_{\mu}\gamma_{\nu}\gamma_{\rho}\gamma_{\sigma}],\\
&[t,ps]_{\rho\sigma}=-[ps,t]_{\rho\sigma} &&=(k_{\alpha}k'_{\beta})\tfrac{1}{4}S_{p}[\gamma_{5}\gamma_{\rho}\gamma_{\sigma}\gamma_{\alpha}\gamma_{\beta}],
\end{aligned} \tag{3.4}$$

$$
\begin{aligned}
&[t,pv]_{\rho\sigma,\mu}=[pv,t]_{\mu,\rho\sigma} &&=im(k-k')_\alpha\tfrac{1}{4}S_p[\gamma_5\gamma_\mu\gamma_\rho\gamma_\sigma\gamma_\alpha],\\
&[t,pt]_{\mu\nu,\rho\sigma}=[pt,t]_{\rho\sigma,\mu\nu} &&=(k_\alpha k'_\beta)\tfrac{1}{4}S_p[\gamma_5\gamma_\rho\gamma_\sigma\gamma_\alpha\gamma_\mu\gamma_\nu\gamma_\beta]+m^2\tfrac{1}{4}S_p[\gamma_5\gamma_\rho\gamma_\sigma\gamma_\mu\gamma_\nu],\\
&[ps,ps] &&=-kk'+m^2,\\
&[ps,pv]_\mu=-[pv,ps]_\mu &&=im(k+k')_\mu,\\
&[ps,pt]_{\rho\sigma}=-[pt,ps]_{\rho\sigma} &&=k_\rho k'_\sigma-k_\sigma k'_\rho,\\
&[pv,pv]_{\mu,\nu} &&=k_\mu k'_\nu+k_\nu k'_\mu+\delta_{\mu\nu}(kk'+m^2),\\
&[pv,pt]_{\mu,\rho\sigma}=-[pt,pv]_{\rho\sigma,\mu} &&=im[(k-k')_\rho\delta_{\mu\sigma}-(k-k')_\sigma\delta_{\mu\rho}],\\
&[pt,pt]_{\mu\nu,\rho\sigma} &&=-(k_\alpha k'_\beta)\tfrac{1}{4}S_p[\gamma_\alpha\gamma_\mu\gamma_\nu\gamma_\beta\gamma_\rho\gamma_\sigma]+m^2\tfrac{1}{4}S_p[\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma]\,.
\end{aligned}
$$

In the above equations, there are eight terms including the ambiguous one, i. e. $[s,s]$, $[v,v]$, $[t,t]$, $[t,pt]$, $[pt,t]$, $[ps,ps]$, $[pv,pv]$ and $[pt,pt]$, and the terms for which we can anticipate the results (uniquely) by the formal requirements, such as the divergence theorem (including the gauge invariancy) and the equivalence theorem, are only two, i. e. $[v,v]$ and $[pv,pv]$.

Further, it will be remarked that the equivalence theorem between $[ps,pv]$ and $[ps,ps]$ does not hold according to the general consideration. However, both requirements are only conditionally satisfied by $[v,v]$ and $[pv,pv]$, while they are exactly satisfied by the other matrix elements. Accordingly, it is hoped that the results of these two terms calculated by using our procedure (2.6) are satisfied exactly, and if these would be surely done, it seems to be naturally accepted that the other matrix elements are also evaluated by the similar procedure.

In fact, after the transformation of moment such as $k=q+\frac{1-v}{2}p$ and $k'=-q+\frac{1+v}{2}p$, by which we get the equation

$$
\begin{aligned}
\frac{\delta(k^2+m^2)}{k'^2+m^2}+\frac{\delta(k'^2+m^2)}{k^2+m^2}&=-\frac{1}{2}\int_{-1}^{1}dv\delta'\left[\frac{1+v}{2}k^2+\frac{1-v}{2}k'^2+m^2\right]\\
&=-\frac{1}{2}\int_{-1}^{1}dv\delta'\left[q^2+\frac{1}{4}(1-v^2)p^2+m^2\right], \qquad (3.5)
\end{aligned}
$$

and we have

$$
\begin{aligned}
&[s,s] &&=\frac{3}{4}[(1-v^2)p^2+4m^2],\\
&[v,v]_{\mu,\nu} &&=(p_\mu p_\nu-\delta_{\mu\nu}p^2)\frac{1-v^2}{2},\\
&[v,t]_{\mu,\rho\sigma}=-[t,v]_{\rho\sigma,\mu} &&=-im[p_\rho\delta_{\mu\sigma}-p_\sigma\delta_{\mu\rho}],\\
&[v,pt]_{\mu,\rho\sigma}=-[pt,v]_{\rho\sigma,\mu} &&=-im\,p_\alpha\frac{1}{4}S_p[\gamma_5\gamma_\rho\gamma_\sigma\gamma_\mu\gamma_\alpha],\\
&[ps,ps] &&=-\frac{1}{4}[3(1-v^2)p^2+4m^2], \qquad (3\text{-}6)
\end{aligned}
$$

$$
\begin{aligned}
[ps, pv]_\mu = -[pv, ps]_\mu \quad &= imp_\mu, \\
[pv, pv]_{\mu,\nu} \quad &= (p_\mu p_\nu - \delta_{\mu\nu} p^2)\frac{1-v^2}{2} - 2\delta_{\mu\nu} m^2, \\
[t, t]_{\mu\nu,\rho\sigma} \quad &= \frac{1-v^2}{2}[p_\mu p_\rho \delta_{\nu\sigma} + p_\nu p_\sigma \delta_{\mu\rho} - p_\mu p_\sigma \delta_{\rho\nu} - p_\nu p_\rho \delta_{\mu\sigma}] \\
&\qquad - \left(\frac{1-v^2}{4} p^2 + m^2\right)(\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho}), \\
[t, pt]_{\mu\nu,\rho\sigma} = [pt, t]_{\rho\sigma,\mu\nu} \quad &= \frac{1-v^2}{2}[p_l p_\mu \delta_{m\nu} + p_m p_\mu \delta_{l\nu} - p_m p_\mu \delta_{l\nu} - p_l p_\nu \delta_{m\mu}] \\
&\qquad - \left(\frac{1-v^2}{4} p^2 + m^2\right) \times (\delta_{l\mu}\delta_{m\nu} - \delta_{l\nu}\delta_{m\mu})
\end{aligned}
$$

where we use $(\gamma_s \gamma_\rho \gamma_\sigma) \equiv (\gamma_l \gamma_m)$, and

$$
\begin{aligned}
[pt, pt]_{\mu\nu,\rho\sigma} \quad &= -\frac{1-v^2}{2}[p_\mu p_\rho \delta_{\nu\sigma} + p_\nu p_\sigma \delta_{\mu\rho} - p_\mu p_\sigma \delta_{\rho\nu} - p_\nu p_\rho \delta_{\mu\sigma}] \\
&\qquad + \left(\frac{1-v^2}{4} p^2 - m^2\right)(\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho}).
\end{aligned}
$$

It is easily seen that our results exactly satisfy the above mentioned requirements and that these results are the most adequate ones in the regions of our considerations. Although these matrix elements include the logarithmic divergences, how to treat these difficulties, that is, whether we subtract divergences as a whole or not, is still questionable, but in the present stage we are obliged to rely on the phenomenological consideration. As far as the self-energies of boson concerned, we already discussed some types, such as $[s, s]$, $[ps, ps]$, $[ps, pv]$, and $[pv, pv]$, in other places.[6)]

§ 4. Removal of Ambiguous Terms. Case 11. A→B+C ($A^\pm \to B^\pm + C$ and $A^0 \to B^0 + B^0$)

As discussed in § 2, the reason why one can not get rid of the difficulties in this case is that the indefinite natures of the integrals are out of the question. That is, the most parts of the results are determined with the mathematical uniqueness and therefore we can not remove the difficulties by the mathematically reasonable techniques. If we apply the regulator method to these cases, we have not the consistent conditions throughout the processes and it will be very doubtful whether the results which are calculated by using this procedure are reliable or not.

Consequently, we here try to examine the validity of the mathematically incorrect identity

$$
\int L\delta''(q_\lambda^2 + L)dq = 0, \tag{4.1}
$$

and then consider at what degrees this procedure determines the results uniquely.*

According to the elaborate calculations by Fukuda, Hayakawa and Miyamoto,[7] we take as the fundamental formulae,

$$\langle[\psi^{+}\gamma_A\tau_A\psi]\rangle_2 = g_B g_C \int dX' dX'' U_B(X') U_C(X'') K_{BCA}(X-X', X''-X), \qquad (4.2)$$

$$K_{BCA}(\xi, \eta) = \frac{1}{2}\left\{1+(-1)^{N_V N_T + N_S}\right\} S_p\left(\sum \bar{S}(\xi)\gamma_B \bar{S}(-\xi-\eta)\gamma_C S^{(1)}(\eta)\gamma_A\right)$$

$$= \frac{2}{(2\pi)^{11}}\left\{1+(-1)^{N_V+N_T+N_S}\right\}\int dk dk' dk'' e^{i(k'+k)\xi + i(k''+k)\eta} K_{BCA}(k, k, k) \qquad (4.3)$$

$$\times\left[\sum \frac{\delta(k^2+m^2)}{(k'^2+m^2)(k''^2+m^2)}\right],$$

$$K_{BCA}(k, k', k'') = \frac{1}{4} S_p[(i\gamma k' - m)\gamma_B(-i\gamma k - m)\gamma_C(i\gamma k'' - m)\gamma_A], \qquad (4.4)$$

and translating it to momentum space, we have

$$\sum \frac{\delta(k^2+{}^2m)}{(k'^2+m^2)(k''^2+m^2)} = \frac{1}{2}\int_{-1}^{1} dv \int_0^1 u du \delta''[l_\lambda^2 + a(1-a)p_\lambda^2 + b(1-b)q_\lambda^2$$

$$-2ab(p_\lambda q_\lambda) + m^2], \qquad (4.5)$$

where

$$k = l + ap + bq \qquad\qquad a = \frac{1+v}{2}u$$
$$k' = -l + (1-a)p - bq \qquad \text{and}$$
$$k'' = -l - ap + (1-b)q \qquad\qquad b = \frac{1-v}{2}u.$$

A) γ-Decay of Neutral Meson.

In this case, we especially take $\gamma_B = \gamma_\mu$, $\gamma_C = \gamma_\nu$ and $p_\lambda^2 = q_\lambda^2 = 0$, and utilize the identity

$$\int [m^2 - 2ab(p_\lambda q_\lambda)]\delta''[l_\lambda^2 + m^2 - 2ab p_\lambda q_\lambda] dl = 0\,. \qquad (4.1a)$$

(1) *Case of Scalar Meson* ($\gamma_A = 1$)

After some calculations, we have

$$K[v, v, s]_{\mu,\nu} = m[-\delta_{\mu\nu}(m^2 - 2abpq) + 2a(2a-1)p_\mu p_\nu + 2b(2b-1)q_\mu q_\nu$$

$$+ (1-2(a+b)+4ab)p_\mu q_\nu + (-1+4ab)(p_\nu q_\mu - \delta_{\mu\nu}pq)],$$

and then if we drop the first terms by using (4.1a) and the second and the third terms by virture of integral of u and v, we get

$$K[v, v, s]_{\mu,\nu} = m[(1-2(a+b)+4ab)p_\mu q_\nu + (-1+4ab)(p_\nu q_\mu - \delta_{\mu\nu}pq)].$$

This results undoubtedly satisfies the gauge invariance and by using the Lorentz

---

* It is remarked that we must only use the procedure (4.1) at the first terms of matrix elements but not the second and following terms by the iterations of L.

condition we have the most reasonable results obtained by Fukuda and Miyamoto, and Steinberger.[8)]

$$\langle[\psi^+\psi]\rangle_2 = -\frac{e^2}{8\pi^2} m \int_{-1}^{1} dv \int_0^1 udu \frac{1-(1-v^2)u^2}{m^2-\frac{1-v^2}{4}u\Box^2} \frac{1}{2} F_{\mu\nu}^2. \tag{4.6}$$

(2) *Case of Pseudoscalar Meson with Both Couplings.*

Following the same calculations as above and using the identity (4.1a), we have

$$\langle[\psi^+\gamma_5\psi]\rangle_2 = \frac{e^2}{4\pi^2}\frac{1}{m}\int_{-1}^{1} dv \int_0^1 udu \frac{\frac{1-v^2}{4}u^2\Box^2}{m^2-\frac{1-v^2}{4}u^2\Box^2}(F_{12}F_{34}+F_{23}F_{14}+F_{31}F_{24}), \tag{4.7}$$

and

$$\langle[\psi^+\gamma_5\gamma_k\psi]\rangle_2 = \langle[\psi^+\gamma_l\gamma_m\gamma_n\psi]\rangle_2$$
$$= \frac{e^2}{8\pi^2}\int_{-1}^{1} dv \int_0^1 udu \frac{(1-v^2)u^2}{m^2-\frac{1-v^2}{4}u\Box^2} \sum_{\substack{s=1 \\ l,m,n\ \mathrm{cycl}}}^{4} F_{sl}\partial_s F_{mn}, \tag{4.8}$$

which not only satisfy the gauge invariancy, but also the equivalence theorem, i.e.

$$\sum \partial_k \langle[\psi^+\gamma_5\gamma_k\psi]\rangle_2 = -2m\langle[\psi^+\gamma_5\psi]\rangle_2. \tag{4.9}$$

B) Decay of τ-Meson into Two π-Mesons.

These problems have been already discussed throughly by Fukuda, Hayakawa and Miyamoto,[7)] Ozaki, Ōneda and Sasaki.[9)] Here, we only try to examine the validity of our procedure for a few examples.

(1) *Decay of Scalar τ-Meson with Scalar Coupling into Two π-Meson with Vector Coupling.*

The calculations can be taken in analogous way as γ-decay and then using the identity

$$\int[m^2-2abpq+a(1-a)p^2+b(1-b)q^2]\delta''(l^2+m^2-2a^bpq$$
$$+a(1-a)p^2+b(1-b)q^2)dl=0, \tag{4.1b}$$

we obtain

$$K[v, v, s]_{\mu,\nu} = m[(1-2(a+b)+4ab)p_\mu q_\nu + (-1+4ab)(p_\nu q_\mu - \delta_{\mu\nu}pq)]. \tag{4.10}$$

If the vector couplings of both π-mesons are those of vector π-mesons, we have

$$K[v, v, s]_{\mu,\nu} = m(-1+4ab)(p_\nu q_\mu - \delta_{\mu\nu}pq), \tag{4.11}$$

while if one of them is the vector coupling of scalar π-meson, the divergence theorem holds, that is,

$$p_\mu K[v, v, s]_{\mu,\nu} = q_\nu K[v, v, s]_{\mu,\nu} = 0. \tag{4.12}$$

However, if both couplings are those of scalar π-mesons with masses μ_+ and μ_0 respectively, we have non-vanishing result

$$p_\mu p_\nu K[v, v, s]_{\mu,\nu} = m\mu_+^2\mu_0^2(-1+4ab), \tag{4.13}$$

and the divergence theorem does not hold in this case. This is very puzzling result and this circumstance indicates that our procedure is not so appropriate, but in the cases of above three types this term being zero, the difficulty will probably not arise.

(2) *Decay of Pseudoscalar τ-Meson with Both Coupling into Meson with Pseudovector Couplings.*

Also in this case the calculations can be accomplished straightforwardly in analogous way as γ-decay, but for the sake of simplicity, we take $\mu_+ = \mu_0 = 0$, though we are afraid of the loss of generality. The results are

$$\langle[\psi^+\gamma^2\psi]\rangle_2 = -\frac{\tilde{f}\tilde{f}_0}{4\pi^2}\frac{1}{m}\int_{-1}^{1} dv\int_0^1 udu\frac{1-2u}{m^2-\frac{1-v^2}{4}u^2\Box^2}\frac{1-v^2}{4}u^2\Box^2$$

$$\times\frac{1}{2}\Big[U_{14}^+U_{23}^0 + U_{12}^+U_{34}^0 + U_{31}^+U_{24}^0 + U_{23}^+U_{14}^0 + U_{34}^+U_{12}^0 + U_{24}^+U_{21}^0\Big], \tag{4.14}$$

and

$$\langle[\psi^+\gamma_5\gamma_k\psi]\rangle_2 = -\frac{\tilde{f}\tilde{f}_0}{8\pi^2}\int_{-1}^{1} dv\int_0^1 udu\frac{(1-v^2)u^2}{m^2-\frac{1-v^2}{4}u^2\Box^2}$$

$$\times\frac{1}{2}\Big[(1-2u)\sum_{l,m,n,s=1}^{4}(U_{sl}^+\partial_s U_{mn}^0 + U_{sl}^0\partial_s U_{mn}^+) + 2(1-u)\times$$

$$\times\sum_{s=1}^{4}(\partial_l U_s^+\cdot\partial_s U_{mn}^0 + \partial^l U_s^0\cdot\partial_s U_{mn}^+)\Big]. \tag{4.15}$$

Accordingly, if we make use of the identity such as

$$\sum_k\partial_k\Big(\sum_{\substack{s=1\\ l,m,n}}^{4}\partial_l U_s^0\cdot\partial_s U_{mn}^+ + \sum_{\substack{s=1\\ l,m,n}}^{4}\partial^l U_s^+\cdot\partial_s U_{mn}^0\Big) = 0, \tag{4.16}$$

then we have easily

$$\sum_k\partial_k\langle[\psi^+\gamma_5\gamma_k\psi]\rangle_2 = -2m\langle[\psi^+\gamma_5\psi]\rangle_2, \tag{4.17}$$

which is the equivalence theorem.

§ 5. Concluding Remarks.

As above discussed, although we could not remove the ambiguity in the present field theory by the appropriate mathematical modifications consistently, we can clarify where the difficulties of ambiguity appear with the concentrated forms

by using our unsatisfactory proposals. Therefore, if there would exist only one method which is applicable to our equations, such as

$$\int (q_\lambda^2+2L)\,\delta'(q_\lambda^2+L)\,dq=0,$$

and

$$\int L\delta''(q_\lambda^2+L)\,dq=0,$$

we can keep out of the complications and dangers of using the regulator method which necessitates the condition $\sum \frac{c_i}{m_i}=0$, $\sum c_i=0$ and $\sum c_i m_i=0$ for the γ-decay problem and the conditions $\sum c_i = 0$ and $\sum c_i m_i^2 = 0$ for the photon self-energy problem.

However, in the present stage of the field theory, we are obliged to content ourselves with such a very unsafe method.

In conclusions, we express our gratitude to Prof. M. Kobayasi for his interest and encouragement throughout in this work and also thanks to Assistant Profs. T. Miyazima and S. Ozaki and Messrs. H. Umezawa and S. Goto for their valuable discussions.

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2) H. Fukuda, Y. Miyamoto, T. Miyazima, S. Tomonaga, S. Ozaki, S. Ôneda and S. Sasaki: Prog. Theor. Phys. **4** (1949), 477.
3) H. Umezawa and R. Kawabe: ibid. **4** (1949), 423, 443.
D. Feldman: Phys. Rev. **76** (1949), 1369.
J. Rayski: Acta Physica Polonica, **9** (1948), 128.
4) J. Schwinger: Phys. Rev. **76** (1949), 790.
5) G. Wentzel: Phys. Rev. **74** (1948), 1070.
6) Y. Katayama and K. Sawada: Prog. Theor. Phys. **4** (1949), 377.
Y. Katayama: ibid. **4** (1949), 396.
7) H. Fukuda, S. Hayakawa and Y. Miyamoto: ibid. **5** (1950), 283.
8) H. Fukuda and Y. Miyamoto: ibid. **4** (1949), 347.
J. Steinberger: Phys. Rev. **76** (1949), 1180.
9) S. Ozaki, S. Ôneda and S. Sasaki: Prog. Theor. Phys. **4** (1949), 524; **5** (1950), 25, 165.

Note added in proof.

Recently, Mr. Y. Ôishi has examined the nature of ambiguous integral (2.4) and concluded that the third term of the last one ought to be dropped from the covariant point of view. Because, if we stand on this view-point, we may conclude that the last integral of (2.4) should be the even function of a and we may be obliged to remove the last term for this reason. We thank to Mr. Y. Ôishi for his kind advices and discussions.

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On the Nature of τ-Mesons. I.

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The mass, the life and the interaction with nucleons of τ-mesons discovered in photographic plates are qualitatively discussed. The probabilities of $\tau\rightarrow\pi+\pi^0$ and $\tau\rightarrow\pi+\gamma$ for various combinations of their couplings are calculated by Tomonaga-Schwinger formalism. The generalized selection rule for the system composed of many mesons is presented, from which some particular sets of the types of these mesons, for example scalar and pseudoscalar, are admitted to explain the rather long life time of τ-meson, $\sim 10^{-11}$ sec. . The life time for $\tau\rightarrow 3\pi$ decay calculated by our method is obtained as long as 10^{-11} sec. consistent with experiments. This result suggests that the interaction of π-meson with nucleon is considerably weak. Some arguments for and against such presumption are discussed. The varieties of mesons, called varitrons, are classified as π- and μ-types.

Introduction.

Should one doubt, as declared by Oppenheimer,[1] whether the quantum field theory of mesons based on the analogy between the Maxwell field and the Yukawa field has any valid content? There exist, to be sure, many ununderstandable points in meson problems, but how ununderstandable they are will only be manifested by treating the problems from the present notion of mesons. Here one may remember the history of quantum electrodynamics that it had at first seriously been criticized by Oppenheimer and was lately saved by himself by cascade theory. It seems to us, therefore, to be not worthless to treat the disintegration of mesons, because it may be a rather simpler case of meson problems than such as nuclear forces as there enter only elementary particles, on the view point of the present quantum field theory.

The first attempt of us along this line was applied to the problem of the decay of neutral meson.[2),3)] Regrettably, however, the result contained some ambiguities, one being the remaining of non-gauge covariant terms and the other the invalidity of the equivalence theorem between the pseudoscalar and pseudovector couplings for the pseudoscalar meson field, in spite of the refinement of the formalism by Tomonaga and Schwinger. The same conclusion was obtained by

1) J. R. Oppenheimer, Rev. Mod. Phys. **21** (1949), 181.

2) H. Fukuda and Y. Miyamoto, Prog. Theor. Phys. **4** (1949), 347, quoted as A.

3) H. Fukuda and Y. Miyamoto, Prog. Theor. Phys. **4** (1949), 392.

Ozaki et al.[4] and lately by Steinberger.[5] Looking for such ambiguity for a while, and adopting only the gauge invariant terms, we get the life time of neutral meson, consistent with the interpretation of cosmic ray phemomena[6] and Berkeley experimant.[7] Also in the case of τ-meson we shall, presumably, obtain any precise knowledge about its life time. If this result could give the explanation of experiments, we might believe the validity of the present quantum field theory of mesons within some limitation, or at least believe that the present theory is the good correspondence to the correct theory in future. It is likely the case, as will be seen in what follows.

In part I we discuss the experimental evidences of τ-meson, in order to settle its necessary features for later development.[8] Parts II and III are devoted to treat the decay modes for $\tau\rightarrow\pi+\pi^0$ and $\tau\rightarrow\pi+\gamma$, respectively, where the selection rule for the meson problem is described based on Schwinger's formalism containing some improvement.[9]

In part IV the decay probability for $\tau\rightarrow3\pi$, which was found experimentally, is calculated. In part V we consider some properties of π-mesons and attempt to classify the varieties of mesons called varitrons under the light of our calculation and other consideration.

I. Experimental evidence.

§ 1. The mass and the life of τ-mesons.

We call here such a kind of a meson as a τ-meson that it has the mass about $900\sim1000$m, m the mass of electron, and gives rise to a large nuclear

4) S. Ozaki, S. Ôneda and S. Sasaki, Prog. Theor. Phys. **4** (1949), 524 and **5** (1950), 25.

5) J. Steinberger, Phys. Rev. **76** (1949), 1180.

6) Y. Fujimoto and S. Hayakawa, Prog. Theor. Phys. **4** (1949), 502, about the bursts under a thick shield. J. Nishimura and S. Hayakawa, ibid. 598, about the intensity of the soft component in the stratosphere. Y. Fujimoto and S. Hayakawa, ibid. **5** (1950), 144, about the bursts produced by a lead plate in high altitudes. S. Hayakawa, ibid. **5** (1950), 158, about the excess electrons in the lower atmosphere. But the statement of one of us (S. Hayakawa, Prog. Theor. Phys. **4** (1949), 386, about the life time of neutral meson may be revised. Accounting for the new evidence found in a very large cosmic ray star, the life of neutral meson may be shorter than 10^{-13} sec.

(M. F. Koplau, B. Peters and H. L. Bradt, Phys. Rev. **76** (1949), 1735; R. F. Marshak, ibid. 1736). If this value of the life time is correct, the prescription concerning the ambiguity appeared in the γ-decay of pseudoscalar meson may be settled as follow. Only the non-gauge invariant terms are eliminated from the result and the equivalence theorem would be given up though it should hold. Then the first term in the expression of the decay probability for pseudoscalr coupling (A (2.17)) is remained and gives the life time as short as 10^{-16} sec. as adopted by Steinberger. Such prescription is, of course, only tentative and not solve the ambiguity.

7) R. Bjorklund, W. Crandall, B. Moyer and H. York, Phys. Rev. **77** (1950), 213.

8) H. Fukuda, S. Hayakawa and Y. Miyamoto, Prog. Theor. Phys. **4** (1949), 388.

9) H. Fukuda and Y. Miyamoto, Prog. Theor. Phys. **4** (1949), 398.

disintegration in Paris photograph[10] or a characteristic disintegration into three lighter mesons in Bristol photograph.[11]

The mass of τ-meson in Paris photograph is estimated as at least 700m, considering the energy consumed by a star production with six prongs containing a σ-meson.

If we take into consideration the fact that more neutrons are evapolated by star production than protons, the mass is not inconsistent with 1000m. This interpretation, that the whole rest energy of a τ-meson vanishes and is transported into nuclear particles and a σ-meson, means that the τ-meson may directly interact with a nucleon like π-meson. Further this τ-meson has apparently negative charge because it is captured by a nucleus before disintegration. There occurs the competition between capture and decay as in the case of μ-meson. As for the capture from the ground state in the atom there may be no trouble as far as the coupling of the τ-meson with nucleon is not weaker than that of μ-meson, although we must pay attention to the fact that the K-orbit of τ-meson lies in the nucleus,[12] which is the case in the star under consideration since the star with five prongs should take place in a Ag or Br nucleus in the photographic emulsion. The main time is spent during the passage to the track end or falling down the energy levels in the crystal of $AgBr$. If the trapping took place in the latter case, the life should be as long as 10^{-4} sec.[13] But we consider in Paris τ-meson that the trapping does not take place and the time spent in the crystal is assumed as about 10^{-12} sec.[14]

Thus the life time of τ-meson must be longer than this value. The track length of the τ-meson appeared in the photographic plate is about $200\,\mu$ but a longer track is obtained in Bristol photograph, which is more legitimate to estimate the life of τ-meson.

Bristol photograph shows a track as long as $3000\,\mu$ which disintegrates into a σ-meson and two fast mesons with ordinary mass. The latter two are positive because the parent τ-meson is considered to be positive unless too long time is spent by any mechanism such as trapping. Then the two fast mesons are unlikely μ-mesons, since μ-meson interacts with nucleon very much weaker than π-meson and the change of the change between two rather strongly coupled mesons, τ and π, will hardly take place.[15] We may, therefore, consider the observed disintegration as $\tau^+ \rightarrow \pi^+ + \pi^+ + \pi^-$. Then the mass of the τ-meson is most reliably

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10) L. Leprince-Ringuet, Hoang Tchang-Fong, L. Janean, and D. Morellet. C. R. **226** (1948), 1897; L. Leprince-Reinguet, Rev. Mod. Phys. **21** (1949), 42.

11) R. Brown, U. Camerini, P. H. Fowler, H. Muirhead, C. F. Powell and D. N. Riston, Nature **163** (1949), 48, 82.

12) Y. Fujimoto, S. Hayakawa and Y. Yamaguchi, Prog. Theor. Phys. **4** (1949), 575.

13) R. Huby, Phil. Mag. **40** (1949), 685.

14) E. Fermi and E. Teller, Phys. Rev. **72** (1949), 399. R. L. Rosenberg, Phil. Mag. **40** (1949), 759.

15) H. Yukawa, Rev. Mod. Phys. **21** (1949), 474.

determined by the conservation law of energy and momentum as 985 m provided the mass of π-meson is 286 m. If we take the mass of π-meson as 275 m,[16] this figure is revised as 950 m. These values of the mass are consistent with that derived from the grain density-range relation and the scattering. In later calculation of life times we may adopt the masses of τ- and π-mesons of 1000 m and 300 m without introducing any serious error.

The life of the τ-meson must be long enough to survive during the flight of longer than 3000 μ. The relation between the life and the track length is considered in what follows as in the case of μ-meson.[17] A meson with mass μ and velocity vc loses its energy per length dl as

$$\frac{d}{dl}(\mu c^2/\sqrt{1-v^2})=a/v^2, \tag{1.1}$$

and per time dt as

$$\frac{d}{dt}(\mu c^2/\sqrt{1-v^2})=ac/v, \tag{1.2}$$

where a represents the loss of energy per unit length when $v\approx 1$. This meson passes the track as long as

$$l=\frac{\mu c^2}{a}\left\{\frac{v^2}{\sqrt{1-v^2}}+2\sqrt{1-v^2}-\frac{v_0^2}{\sqrt{1-v_0^2}}-2\sqrt{1-v_0^2}\right\}, \tag{1.3}$$

and spends the time as

$$t=\frac{\mu c}{a}\left\{\frac{v}{\sqrt{1-v^2}}-\mathrm{Sin}^{-1}v-\frac{v_0}{\sqrt{1-v_0^2}}+\mathrm{Sin}^{-1}v_0\right\}, \tag{1.4}$$

before slowed down to the velocity v_0. At the track end the terms containing v_0 can be neglected where v_0 is comparable to the velocity of atomic electrons in outer shell. For the nonrelativistic velocity, $v\ll 1$, (1.3) and (1.4) are approximated by

$$l=\frac{\mu c^2}{a}\frac{v^4}{4}(1+v^2+\cdots), \tag{1.3'}$$

and

$$t=\frac{\mu c}{a}\frac{v^3}{3}(1+\frac{9}{10}v^2+\cdots). \tag{1.4'}$$

If we take $a=8$ MeV cm^{-1}, accounting for the density of Kodak NT4 plate as 4 g cm^{-3}, the kinetic energy and the velocity at 3000 μ before the track end are estimated as about 35 MeV and about 0.4c.

The time spent in the photographic plate is thus obtained from (1.3′) and (1.4′) as

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16) F. M. Smith, W. H. Barkas, H. Bradner and F. Gardner, unpublished.

17) H. Yukawa and T. Okayama, Sci. Pap. I. R. C. R. **36** (1939), 385.

$$t = \frac{3}{4}\frac{l}{vc} \sim 3 \times 10^{-11} \text{ sec.} \tag{1.5}$$

The real life is longer than this figure.

Above estimation is rather uncertain because it stands only one track. There seems to be one more track similar to $\tau \rightarrow 3\pi$ decay obtained by Armenian group.[18] The masses of a parent and a descendent, $750 \sim 800$ m and $180 \sim 200$ m, are slightly lower than those of Bristol. There may, however, be possible that the mass determination by grain count undergoes some systematic error, such as in the case of the original work about π-meson.[19] If this decay could be identified with $\tau \rightarrow 3\pi$, the track length of the parent, $1600\,\mu$, would give a further information about the life of τ-meson. Accordingly, we may adopt the life time longer than 10^{-11} sec.

Furthermore, if the Armenian decay were established as $\tau \rightarrow 3\pi$, the mode of disintegration into three π-mesons would be wholely symmetrical over π-mesons irrespective of the sign of their charges. For the observed π-meson with lower energy is positive because any nuclear disintegration is not found at its end, while two faster mesons are positive and negative, in contrast to Bristol decay.

Another evidence discussed in our previous note, the cloud chamber photographs obtained by Rochester and Butler,[20] is not necessary identified with τ-mesons by the reason discussed in V. In later treatment we will not consider this evidence as due to τ-mesons.

§ 2. The production of τ-meson.

The magnitude of the interaction of τ-meson with nucleons is inferred by considering its production, assuming that a π-meson has the same nature as τ-meson and the only difference between both mesons lies in the magnitude of their masses and coupling constants.

The abundance of τ-mesons relative with π-mesons is obscure because of too poor data. Only datum available in our hands is the frequency of occurrence of τ- and π-mesons in Bristol experiment.[11] They obtained only one τ-meson in their photographic plates at hight 6.9 m H_2O, while 30 π^+ and 30 π^--mesons are observed at the same time. Then the abundance ratio between τ- and π-mesons may be estimated as about 1%. The frequency of mesons is dependent on the intensity of ancestor rays, the production rate of mesons, their mean range and their stopping rate. But the product of the former two quantities are considered to be essential, since the latter two may compensate with each other. Accordingly, we have only to discuss the various types of production mechanism and their cross sections.

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18) A. I. Alikhaniyan, E. M. Samoilovich, I. I. Gurevich, and Kh. P. Babayan, J. Exp. Theor. Phys. **19** (1949), 667.
19) C. M. G. Lattes, G. P. S Occhialini and C. F. Powell, Nature, **160** (1949), 453, 487.
20) G. D. Rochester and C. C. Butler. Nature, **160** (1947), 855.

The production cross section for τ-meson is different from that of π-meson by the large mass and the weak coupling with nucleons. And we must take into consideration the electromagnetic effect for its production process if its nuclear interaction is too weak. Therefore, these possible mechanisms, the pair creation in nuclear Coulomb field, the photomesonic effect and the nuclear process by nucleon-nucleon collision, are compared as follows.

1. *Pair creation by a photon in nuclear Coulomb field.* The production cross section per nucleus with charge Ze for this process is given by[21)]

$$\sigma_1 \approx \frac{5}{9} \alpha Z^2 \left(\frac{e^2}{\mu_\tau c^2} \right)^2 = 3.2 \times 10^{-34} Z^2 \text{ cm}^2, \tag{1.5}$$

where μ_τ represents the mass of τ-meson and α the fine structure constant 1/137. The cross section is further dependent on the energy of an incident photon and the dependency is largely affected by the types of meson field. But here we neglect such a factor, since we only concern the relative magnitude for three processes. This cross section does not depend on the nuclear interaction of τ-meson and amounts as large as 2×10^{-30} cm^2 for *Pb*.

2. *Photo-mesonic effect.* The cross section per nucleus that, for example, a photon is absorbed by a proton and the latter emits a τ-meson transmuting into a neutron, is given by[22)]

$$\sigma_2 \approx \pi A \left(\frac{Ge}{\mu_\tau c^2} \right)^2 = 3.4 \times 10^{-32} A \text{ for } G^2 = 10^{-3}. \tag{1.6}$$

where G means the dimensionless coupling constant of a τ-meson with nucleon and A the mass number of the collided nucleus. This process is more effective than the pair creation even in *Pb* as far as $G^2 \gtrsim 3 \times 10^{-4}$. The photomesonic effect would be the main source of τ-meson if the ancestor rays were photons. But it may be unlikely the case, because it results in too few τ-mesons in comparison with π-mesons, which are mainly produced by the nucleon component.

3. *Nucleon-nucleon collision.* By a nuclear collision not only the π-meson field but also the τ-meson field is excited due to the motion of spin and isotopic spin of nucleon by nuclear force, to which π-mesons are mainly responsible. Thus the production ratio of τ- to π-mesons is the ratio of both coupling constants G^2/g^2, where g means the dimensionless coupling constant of π-meson with a nucleon. Since the cross section for the nuclear collision of a nucleon is, as usual, given by the geometrical cross section of a nucleus, the production cross section for τ-meson is

$$\sigma_3 \approx \pi r_0^2 A^{2/3} G^2 = 6.2 \times 10^{-29} A^{2/3} \text{ cm}^2, \text{ for } G^2 = 10^{-3}. \tag{1.7}$$

where $r_0 A^{1/3}$ is the radius of a nucleus with mass number A.

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21) F. Booth and A. H. Wilson, Proc. Roy. Soc. **157** (1940), 483.

22) M. Kobayasi and T. Okayama, Proc. Phys. Math. Soc. Japan, **21** (1939), 1.

Comparing σ_3 with σ_2, σ_3 is always larger than σ_2. Thus we infer that the nucleon-nucleon collision is most effective for τ-meson production.

In order to obtain τ-π ratio, we must consider that the threshold energy for the τ-meson production is about three times larger than that of the π-meson. The intensities of parent nucleons are compared in the neighbourhood of the threshold energies for both mesons, since the mesons observed in photographic plates are sufficiently slow. Then the intensity of nucleons for τ-meson production is about one tenth of that for π-meson, considering the power energy spectrum of the neucleon component. But the energy range responsible to the production of mesons observed may be larger for τ-meson. Since we can not correctly take into consideration such effect, we assume tentatively the abundance ratio as G^2/g^2. The fact that τ-mesons are about one percent of π-mesons leads us the estimation as $G^2 \approx 10^{-3}$, provided $g^2 = 10^{-1}$.

In the following calculation we shall use the value $G^2 = 10^{-3}$ tentatively. The more accurate determination of G should refer to the life time of τ-meson (IV).

II. The Decay of $\tau \rightarrow \pi + \pi^0$.

In what follows we calculate the decay probability of τ-meson for three possible modes. We assume, as related in introduction, that both π- and τ-mesons are Bosons obeying Yukawa equation and protons and neutrons are Fermions obeying Dirac equation. Further we refer to the Tomonaga-Schwinger's formalism of the quantum field theory, though including some ambiguity. Since we can not know which of types of meson fields must be taken, or the type should be determined by comparing the result of our calculation with experiment, the calculation is carried out for whole possible type of couplings. But we need not calculate all of them, as the procedure is much simplified by considering some general rules. The method of calculation is described in what follows, referring Schwinger's method[24] and its improvement. The same notations are used as Schwinger and A as far as possible.

§ 1. Fundamental formalism.

The equation describing a system containing τ-$^+$, π^+- and π^0-mesons and nucleons is expressed on the basis of Tomonaga-Schwinger's formalism as follows.

$$\left\{\varphi^+\gamma_L\tau_L\varphi U_L + \varphi^+\gamma_M\tau_M\varphi U_M + \varphi^+\gamma_N\tau_N\varphi U_N + \frac{\delta}{i\delta C}\right\}\bar{\Psi}[C]=0, \tag{2.1}$$

where we use hereafter the unit of $\hbar = c = 1$, and the coupling constants between mesons and nucleons are dropped for a while for convenience. $\Psi[C]$ is the

23) H. Fukuda and Y. Miyamoto, Prog. Theor. Phys. in press.

24) J. Schwinger, Phys. Rev. 74 (1948), 1439; 75 (1949), 651; 76 (1949), 790.

Schrödinger functional on a variable surface C and $\partial/\partial C$ means the functional derivative with respect to this surface. φ means the spinor describing the nucleon and φ^+ is connected with the hermitic conjugate φ^* of φ by $\varphi^+=\varphi^*\gamma_4$. U_L, U_M, U_N represent the potentials of meson fields or their space-time derivatives, which are specified by suffices L, M and N corresponding to $\pi^\pm$, π^0- and τ^+-mesons. γ_L *etc.* and τ_L *etc.* mean the spin and isotopic spin operators, respectively. Corresponding to scalar, vector, tensor, pseudovector or pseudoscalar coupling, γ_L represents 1, γ_i, γ_{ij}, γ_{ijk} or γ_5, where i and j indicate the components of space-time, 1 to 4. Isotopic spin τ is specified according to the problem under consideration, for example

$\tau_L=\tau_{NP}$ for U_L containing an operater to create a π^--meson,

$\tau_N=\tau_{PN}$ for U_N containing an operator to annihilate a τ^+-meson,

$$\tau_M=\begin{cases}1 & \text{for } U_M \text{ describing a neutral meson of neutral theory,}\\ \tau_3 & \text{for } U_M \text{ describing a neutral meson of symmetrical theory.}\end{cases}$$

In order to get the change of the state vector by vaccum nucleons and $\pi^\pm$- and π^0-mesons when there exists initially only a τ^+-meson, we carry out the contact transformation

$$\Psi[C]=T\Phi, \tag{2.2}$$

where T obeys the equation

$$\left\{\varphi^+\gamma_L\tau_L\varphi U_L+\varphi^+\gamma_M\tau_M\varphi U_M+\frac{\partial}{i\partial C}\right\}T[C]=0, \tag{2.3}$$

Then the fundamental equation (2.1) is reduced to

$$\left\{T^{-1}\varphi^+\gamma_N\tau_N\varphi U_N T+\frac{\partial}{i\partial C}\right\}\Phi=0. \tag{2.1'}$$

Corresponding to the concerning process, we obtain the vacuum expectation value of the transformed Hamiltonian in (2.1′) in the absence of nucleons in initial and final states. The lowest order process $\tau^+\rightarrow\pi^++\pi^0$ is, as shown in Appendix A,[25]

$$\begin{aligned}\langle T^{-1}\varphi^+\gamma_N\tau_N\varphi U_N T\rangle_{2,0}=&\frac{1}{2}\int_{-\infty}^{\infty}dx'\int_{-8}^{\infty}dx''U_L''U_M'U_N\\ &\times\{S_P\sum(\bar{S}(X-X'')\gamma_L\bar{S}(X''-X')\gamma_M S^{(1)}(X'-X)\gamma_N)\tau_{NP}\tau_M\tau_{PN}\\ &+S_P\sum(S^{(1)}(X-X')\gamma_M\bar{S}(X'-X'')\gamma_L\bar{S}(X''-X)\gamma_N)\tau_M\tau_{NP}\tau_{PN}),\end{aligned} \tag{2.4}$$

since the radiative correction is neglected throughout in this work. In (2.4) the functions $\bar{S}$ and $S^{(1)}$ are the same as Schwinger's and

$$\sum\bar{S}\gamma_L\bar{S}\gamma_M S^{(1)}\gamma_N=\bar{S}\gamma_L\bar{S}\gamma_M S^{(1)}\gamma_N+\bar{S}\gamma_L S^{(1)}\gamma_M\bar{S}\gamma_N+S^{(1)}\gamma_L\bar{S}\gamma_M\bar{S}\gamma_N.$$

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25) H. Fukuda and Y. Miyamoto, Prog. Theor. Phys. 4 (1949), 389.

(2.4) reveals the selection rule for the decay process under consideration (c. f.) Appendix A) :

$$\langle T^{-1}\varphi^{+}\gamma_{N}\tau_{N}\varphi U_{N}T\rangle_{2,0}=\frac{1}{2}\{1+(-1)^{N_v+N_t+N_3}\}\int_{-\infty}^{\infty}dx'\int_{-\infty}^{\infty}dx''$$

$$U_{L}''U_{M}'U_{N}S_{P}\{\sum\bar{S}(X-X'')\gamma_{L}\bar{S}(X''-X')\gamma_{M}S^{(1)}(X'-X)\gamma_{N}\}, \qquad (2.4)$$

where N_v, N_t and N_3 represent the numbers of occurrence of vector and tensor couplings and τ_3, respectively. This equation, connected with the features of $\bar{S}$ and $S^{(1)}$, much simplifies our calculation as shown in what follows.

§ 2. The general rules for matrices.

1. *Selectrion rule (generalized Furry's theorem).* From (2.4′) we have the case when the matrix element vanishes:

$$N=N_v+N_t+N_3=\text{odd}, \qquad (2.5)$$

This rule is the generalization of Furry's theorem in the positron theory[26] and the decay of a neutral meson.[27] The rule (2.5) is further specified according to the neutral or symmetrical theory.

(a) Neutral theory. The matrix element vanishes when

$$N_v+N_t=\text{odd}, \qquad (2.6a)$$

because τ_3 is not included in this case. The disintegration process is forbidden if there are odd numbers of vector and tensor couplings.

(b) Symmetrical theory. In this case the coupling of π^0-meson with nucleon is described by τ_3 and, therefore, N_3 is always unity or odd. Then the forbidden condition is given by

$$N_v+N_t=\text{even} \qquad (2.6b)$$

or that the numbers of vector and tensor couplings appear even times in (2.4′).

2. *Lorentz invariancy of interaction Hamiltonian.* The decay process is forbidden in some special combinations of three mesons, because the interaction Hamiltonian composed of them is not Lorentz invariant. As seen in (2.4′), the matrix element under consideration consists in five covariant quantities with respect to Lorentz transfoımation, the wave functions of three mesons, U_L, U_M and U_N, and two independent propagation vector of any two mesons. If, for example, two of these mesons are scalar and one is pseudoscalar, one can never construct a scalar quantity with these meson fields and two energy-momentum vectors. Such circumstances occur in the following sets of mesons.

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26) W. H. Furry, Phys. Rev. **51** (1937), 125.

27) S. Sakata and Y. Tanikawa, Phys. Rev. **57** (1940), 548. see also A.

$$S\,S\,P_S,\ S\,S\,P_V,\ S\,V\,P_S,\ P_S\,P_S\,P_S,\ P_S\,P_S\,P_V. \tag{2.7}$$

These cases are unconditionally forbidden.

3. *Divergence theorem.* The vector coupling of scalar meson vanishes in some cases by so called divergence theorem.[28] For example, in the case of scalar τ-meson with vector coupling (2.4′) is reduced as follows. Firstly by integration by part

$$\int_{-\infty}^{\infty}dx'\int_{-\infty}^{\infty}dx''S_p\{\Sigma\bar{S}(X-X'')\gamma_L\bar{S}(X''-X')\gamma_M S^{(1)}(X'-X)\gamma_i\}U_L''U_M'\partial_i\phi$$

$$=\int_{-\infty}^{\infty}dx'\int_{-\infty}^{\infty}dx''S_p\Sigma\{\gamma_L\bar{S}(X''-X')\gamma_M S^{(1)}(X'-X)\}\{(-\gamma\overrightarrow{\partial}+x)+(\gamma\overleftarrow{\partial}-x)\}$$
$$\bar{S}(X-X'')\}U_L''U_M'\phi. \tag{2.8}$$

where φ represents a scalar meson field instead of U_N, $\overleftarrow{\partial}$ and $\overrightarrow{\partial}$ mean the differentiation respective to space-time coordinates operating to left and right respectively, and x is the mass of nucleon. Making use of the relations

$$\bar{S}(X'-X)(\gamma\overleftarrow{\partial}-x)=(-\gamma\overrightarrow{\partial}-x)\bar{S}(X-X')=\delta(X-X')$$

where $\delta(X)$ is the four dimensional δ-function, and

$$S^{(1)}(X'-X)(\gamma\overleftarrow{\partial}-x)=(-\gamma\overrightarrow{\partial}-x)S^{(1)}(X-X')=0.$$

(2.8) turns into

$$\int_{-\infty}^{\infty}dx'S_p\{\gamma_L\bar{S}(X-X')\gamma_M S^{(1)}(X'-X)+\gamma_L S^{(1)}(X-X')\gamma_M\bar{S}(X'-X)\}$$
$$\times\{U_M'U_L\phi-U_L'U_M\phi\}. \tag{2.8′}$$

Trace in (2.8′) vanishes except for the case where only one of γ_L and γ_M is vector or tensor coupling according to Furry's theorem like as in § 1.

The forbidden cases for the reason are the following sets of couplings:

$$v\,v\,s,\ v\,v\,pv,\ vt\,pv,\ v\,v\,ps,\ v\,t\,ps,\ v\,t\,s,^{*} \tag{2.9}$$

provided that one of vector couplings is of scalar meson.

Besides there are more forbidden cases because the trace of each term (2.8′) vanishes, which are the sets

$$v\,s\,pv,\ v\,s\,ps. \tag{2.10}$$

Such cases do not appear in general in higher order processes, where more numbers of $\bar{S}$ and $S^{(1)}$ are included. In their case, however, the calculation can

28) E. J. Dyson, Phys. Rev. 73 (1948), 929.

* The capital notations, S, V, Pv and Ps, refer to the types of meson fields, *i, e,* scalar, vector, pseudovector and pseudoscalar, while the small letters, s, v, t, pv, ps, represent the types of interactions between mesons and nucleons, scalar, vector, tensor, pseudovector and pseudoscalar, respectively.

be reduced to the case containing only two S by this theorem.

4. *Equivalence theorem.* In some cases, as is well known, the pseudoscalar and pseudovector couplings of a pseudoscalar meson are equivalent.[29] If we know these cases by a general rule, we have only to calculate the matrix element concerning the pseudoscalar coupling alone.

This theorem is proved, for example in the case of pseudoscalar τ-meson, with pseudovector coupling, as follows. Analogous to the case of divergence theorem (2.4′) is transformed as

$$\frac{1}{2}\int_{-\infty}^{\infty}dx'\int_{-\infty}^{\infty}dx''\,[S_p\{\sum\bar{S}(X-X'')\gamma_L\bar{S}(X''-X')\gamma_M S^{(1)}(X'-X)\gamma_5\gamma_i$$
$$+(-1)^{N_3}\sum\bar{S}(X-X')\gamma_M\bar{S}(X'-X''(\gamma_L S^{(1)}(X''-X)\gamma_5\gamma_i\}U_L''U_M'\partial_i\phi\}$$
$$=2\varkappa\langle\gamma_5\rangle+\frac{1}{2}\int_{-\infty}^{\infty}dx'[S_p\sum\bar{S}(X-X')\gamma_L S^{(1)}(X'-X)[\gamma_M,\gamma_5]_\pm U_i'U_N\phi$$
$$-S_p\sum\bar{S}(X-X')\gamma_M S^{(1)}(X-X')[\gamma_5,\gamma_L]_\pm U_L\phi U_M'], \tag{2.11}$$

where $\langle\gamma_5\rangle$ means the transformed expression of the left hand side of (2.11) as $\gamma_5\gamma_i\to\gamma_5$ and $\partial_i\phi\to\phi$, which is the matrix element for pseudoscalar coupling. And $[\gamma_M,\gamma_5]_\pm=\gamma_M\gamma_5\pm\gamma_5\gamma_M$, double sign corresponding to the neutral and symmetrical theory, respectively. The equivalence theorem holds if the second term in the right hand side of (2.11) vanishes. It is shown that this is the case when either of the following conditions holds: Furry's theorem, $[\gamma_M\gamma_5]_\pm=[\gamma_5\gamma_L]_\pm=0$ or trace $=0$. These are the following sets, *s s pv, ps pv pv, ps ps pv, pv pv pv, v v pv* for neutral theory,* *v s pv, t s pv, t ps pv,* for symmetrical theory.* Such cases become rarer for higher order processes. The third order process is advantageous because S functions lie side by side in each other.

5. *Symmetry and Anti-symmetry of matrix element.* As seen from (2.4), the suffices L, M and N are symmetry or anti-symmetry in each other. Therefore, the result is not changed by taking any L, M and N as τ, π and π^0, since if the types and couplings of these meson fields are interchanged the suffices have only to be interchanged correspondingly. For the sake of this character the labour of our calculation is much saved, by interchanging the momenta $(P_\pi, P_\pi{}^0, -P_\tau)$ of three mesons. Or making use of the relation between the momentum P and mass μ, $P_\pi^2=-\mu_\tau^2$, $P_{\pi^0}^2=-\mu_{\tau 0}^2$, $P_\tau^2=-\mu_\tau^2$, we have only to interchange the masses in the last expressions of life times.

Without the help of above general rules, the exhaustive calculation about all types of couplings 1024, will be practically impossible. Such regard is of particular

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29) E. C. Nelson, Phys. Rev. **60** (1946), 830.
F. J. Dyson, Phys. Rev. **73** (1948), 929.
Y. Nambu, Prog. Theor Phys. **3** (1948), 444.
K. M. Case, Phys. Rev. **76** (1949), 14.

* One of *pv* couplings in each set is the coupling of pseudoscalar meson.

importance for higher order processes, such as the disintegration into many particles and the multiple production of mesons.

§ 3. The evaluation of matrix elements.

Accounting for the above five rules, we evaluate the matrix elements for all possible processes. Regrettably, nevertheless, the results are not free from any ambiguity arising from the pathological nature of D-function ; the equivalence and divergence theorems do not hold in the cases where they should hold as in the case of the γ-decay of neutral meson.[2),4),5)] Such a term exists only in the first term in the expansion with respect to the power of the reciprocal mass of nucleon, $1/\varkappa$. It seems to be an advantage of Tomonaga-Schwinger formalism to be able to see where and how exists ambiguity. Then we have two ways to remedy the ambiguity.

One is the method of regulator.[30)] But unfortunately it seems to be irregular, since there is no uniqueness in $\varkappa$-depencence. For example, if we could consider the coupling constant as $g/\varkappa$ instead of g, the first term in the matrix element, proportional to $g\ 1/\varkappa$, should be transformed as to be proportional to $\varkappa^0$. In the latter case this term be dropped by regulater, whereas in the former case the term should be remained. Such situation may be partly due to the nature of the fundamental equations of mechanics, which are not changed if one changes the length and the mass in such a way that they are inversely proportional to each other. Furthermore the ambiguity in regulator is found by several authors.[31)] Accordingly, we make use of regulator as a conventional prescription to remedy the ambiguity.

The other is the method to fill up the ambiguity by other appropriate conditions such as equivalence and divergence. But there are some ambiguities with no relation with those theorems, for which we have not any prescription to do with.

Apart from such ambiguity there remain infinities, which are characteristic for the quantum field theory. Some of them may be dropped by the regulator or by the compensation by adopting the appropriate values of a pair of coupling constants in a meson field. Whether or not such presc̄riptions are justice will be left for future investigations.

Leaving these questions unsolved, we evaluate the matrix elements in what follows, Introducing the Schwinger's representations into $\bar{S}$ and $S^{(1)}$,

$$\bar{S}(X)=\frac{1}{(2\pi)^4}P\int\frac{i\gamma\cdot K-\varkappa}{K^2+\varkappa^2}e^{iKX}\,dK,$$

30) W. Pauli and F. Villars, Rev. Mod. Phys. **21** (1949), 434.

31) Y. Katayama, Prog. Theor. Phys. **4** (1949), 377, H. Fukuda, Y. Miyamoto, T. Miyazima and S. Tomonaga, ibid. (1949), 385. S. Ozaki, S. Oneda and S. Sasaki, Prog. Theor. Phys. **4** (1949), 477, 524. Many authors discussed this problem in the meetings of the Physical Society of Japan in Spring and Autumn of 1949.

and

$$S^{(1)}(X) = \frac{1}{(2\pi)^3}\int (i\gamma\cdot K - x)\,\delta(K^2+x^2)e^{iKX}\,dK,$$

where K^2 is the square of the momentum of a nucleon, (2.4) is represented as

$$\frac{\{1+(-1)^N\}}{2(2\pi)''}\int dK\int dK'\int dK'' S_p\{(i\gamma K-x)\gamma_L(i\gamma K'-x)\gamma_M(i\gamma K''-x)\gamma_N\}$$

$$\times\left\{\frac{\delta(K''^2+x^2)}{(K^2+x^2)(K'^2+x^2)}+\frac{\delta(K^2+x^2)}{(K'^2+x^2)(K''^2+x^2)}+\frac{\delta(K'^2+x^2)}{(K''^2+x^2)(K^2+x^2)}\right\}$$

$$\times\int_{-\infty}^{\infty}dX'\int_{-\infty}^{\infty}dX''\ \exp\ [i\{K'\cdot X-X'')+(K'\cdot X''-X')+(K''\cdot X'-X)$$

$$+P_L\cdot X''+P_M\cdot X'-P_N X\}]\ U_L U_M U_N$$

where P_L, P_M and P_N mean the four dimensional propagasion vectors of $\pi^\pm$, π^0- and τ^+-mesons, respectively, and the fourth component of P is i times positive value. The minus sign before P_N means the annihilation of a τ^+-meson.

There holds a relation

$$\frac{\delta(K''^2+x^2)}{(K^2+x^2)(K'^2+x^2)}+\frac{\delta(K^2+x^2)}{(K'^2+x^2)(K''^2+x^2)}+\frac{\delta(K'^2+x^2)}{(K''^2+x^2)(K^2+x^2)}$$

$$=\frac{\delta(K''^2+x^2)}{(K^2-K''^2)(K'^2-K''^2)}+\frac{\delta(K^2+x^2)}{(K'^2-K^2)(K''^2-K^2)}+\frac{\delta(K^2+x^2)}{(K''^2-K'^2)(K^2-K'^2)}$$

$$=\int_0^1\frac{1}{2}u\,du\int_{-1}^1 dv\ \delta''(x^2+aK^2+bK'^2+cK''^2) \tag{2.12}$$

where

$$a=\frac{1}{2}u(1+v),\quad b=1-u,\quad c=\frac{1}{2}u(1+v),\quad a+b+c=1$$

Substituting (2.13) into (2.12) and integrating over x' and x'', we obtain

$$\frac{\{1+(-1)^N\}}{4(2\pi)^3}\int dK\int dK'\int dK''\delta(K-K'-P_L)\delta(K'-K''-P_M)$$

$$\int_0^1 u\,du\int_{-1}^1 dv\ \delta''(x^2+aK^2+bK'^2+cK''^2)\,U_L U_M U_N$$

$$\times S_p\{(i\gamma\cdot K-x)\gamma_L(i\gamma\cdot K'-x)\gamma_M(i\gamma\cdot K''-x)\gamma_N\} \tag{2.14}$$

Accounting for $P_N=P_L+P_M$ and $a+b+c=1$, we transform K, K' and K'' into $\tilde{K}$, by

$$K=\tilde{K}+bP_L+cP_N\quad K'=\tilde{K}+cP_M-aP_L\quad K''=\tilde{K}-aP_N-bP_M,$$

$$x^2+aK^2+bK'^2+cK''^2=x^2+\tilde{K}^2+abP_L^2+bcP_M^2+caP_N^2$$

Substituting these relations into (2.14), we get

$$\frac{(1+(-1)^N)}{4(2\pi)^3}\int d\tilde{K}\int_{-1}^{1}dv\int_0^1 udu\,\delta''(x^2+\tilde{K}^2+abP_L^2+bcP_M^2+caP_N^2)U_LU_MU_N$$

$$S_p\{i\gamma\cdot(\tilde{K}+bP_L+cP_N)-x\}\gamma_L\{i\gamma\cdot(\tilde{K}+cP_M-aP_L)-x\}\gamma_M\cdot$$

$$\times\{i\gamma\cdot(\tilde{K}-aP_N-bP_M)-x\}\gamma_N] \qquad (2.15)$$

Making use of the formulas, which are proved in Appendix B,

$$\int d\tilde{K}\,\delta''(\tilde{K}^2+A)=\pi/A, \qquad \int d\tilde{K}\tilde{K}_i\delta''(\tilde{K}+A)=0,$$

$$\int d\tilde{K}\,\tilde{K}_i\tilde{K}_j\delta''(\tilde{K}^2+A)=\frac{1}{4}\delta_{ij}\int d\tilde{K}\,\tilde{K}^2\delta''(\tilde{K}^2+A)$$

$$=\frac{\pi}{2}\delta_i\left\{\log\frac{(K+\sqrt{K^2+A})^2}{x^2}-\frac{5}{2}-\log\frac{A}{x^2}\right\}_{K\to\infty},$$

$$\int d\tilde{K}\,\tilde{K}_i\tilde{K}_j\tilde{K}_k\delta''(\tilde{K}^2+A)=0, \qquad (2.16)$$

(2.15) is reduced to

$$\frac{1+(-1)^N}{2(4\pi)^2}\int_0^1 udu\int_1^1 dv\left\{\frac{1}{x^2+abP_L^2+b.P_M^2+caP_N^2}\right.$$

$$\times S_p[\{i\gamma(bP_L+cP_N)-x\}\gamma_L\{i\gamma(cP_M-aP_L)-x\}\gamma_M\{-i\gamma(aP_N+bP_M)-x\}\gamma_N]$$

$$+\left\{-\log\infty+\log\left(1+\frac{abP_l^2+bcP_M^2+caP_N^2}{x^2}\right)\right\}$$

$$S_p[\gamma_N\gamma_L\gamma_M\delta_L\{i\gamma(-aP_N-bP_M)-x\}+\gamma_M\gamma_N\gamma_L\delta_N\{i\gamma(cP_M-aP_L)-x\}$$

$$+\gamma_L\gamma_M\gamma_N\delta_M\{i\gamma(bP_L+cP_M)-x\}] \qquad (2.17)$$

where we use the following abbreviations

$$\log\infty=\left\{\log\frac{(K+\sqrt{K^2+x^2})^2}{x^2}-\frac{5}{2}\right\}_{K\to\infty},$$

$$\gamma_i\gamma_L\gamma_i=2\delta_L\gamma_L,\quad \gamma_i\gamma_M\gamma_i=2\delta_M\gamma_M,\quad \gamma_i\gamma_N\gamma_i=2\delta_N\gamma_N,$$

$$\delta_L=2,\ -1,\ 0,\ 1,\ -2 \text{ for } \gamma_L=1,\ \gamma_i,\ \gamma_{ij},\ \gamma_{ijk},\ \gamma_5 \text{ etc.}$$

$$P_L^2=-\mu_\pi^2,\ P_M^2=-\mu_{\pi 0}^2,\ P_\tau^2=-\mu_\tau^2. \qquad (2.18)$$

In order to evaluate the integral in (2.17), the integrand is expanded by the power of (μ/x). Accounting for

$$\int_0^1 udu\int_{-1}^1 dv\, a^\alpha b^\beta c^\gamma=2.\,\alpha!\ \beta!\ \gamma!/(\alpha+\beta+\gamma+2)! \qquad (2.19)$$

(see Appendix B), the integration is carried out to the second term. If we notice the fact that the trace for the odd number of γ_i vanishes, the result is classified in the following two cases.

a) The number of γ_i in γ_L, γ_M and γ_N is even.

$$(2.17)=\frac{1+(-1)^N}{2(4\pi)^2}x\, U_L U_M U_N S_p\Big[\gamma_L\gamma_M\gamma_N\Big\{(\delta_L+\delta_M+\delta_N)\ \log\ \infty-1$$

$$+(1-\delta_L-\delta_M-\delta_N)\frac{P_L^2+P_M^2+P_N^2}{12x^2}-\frac{\varepsilon_L P_L^2+\varepsilon_M P_M^2+\varepsilon_N P_N^2}{12x^2}\Big\}$$

$$+\frac{1}{12x^2}\Big\{(\gamma P_N)\gamma_L(\gamma P_M)\gamma_M\gamma_N(2-\varepsilon_M-\varepsilon_N)+(\gamma P_L)\gamma_M\gamma P_N\gamma_N\gamma_L(2-\varepsilon_N-\varepsilon_L)$$

$$-(\gamma P_M)\gamma_N(\gamma P_L)\gamma_L\gamma_M(2-\varepsilon_L-\varepsilon_N)\Big\}+0(\mu^4/x^4)\Big] \qquad (2.20)$$

β) The number of γ_i in γ_L, γ_M and γ_N is odd.

$$(2.17)=\frac{1+(-1)^N}{2(4\pi)^2}U_M U_N U_L\, S_p\Big[\frac{1}{3}\Big\{\gamma_L\gamma_M\gamma_N(\gamma\cdot P_L+P_N)(1-\delta_M\ \log\ \infty)$$

$$+\gamma_M\gamma_N\gamma_L(\gamma\cdot P_M-P_L)(1-\delta_L\log\infty)+\gamma_N\gamma_L\gamma_M(\gamma-P_N-P_M)(1-\delta_L\log\infty)$$

$$+\frac{1-\varepsilon_L\varepsilon_M\varepsilon_N}{60x^2}(\gamma P_L)\gamma_L(\gamma P_N)\gamma_M(\gamma\cdot P_N)\gamma_N$$

$$+\frac{1}{60x^2}\left\{\begin{aligned}&\gamma_L\gamma_M\gamma_N[(\gamma P_I)(2P_L^2+2P_M^2+P_N^2)+(\gamma P_N)(P_L^2+2P_M^2+2P_N^2)\}(\delta_M-1)\\&\qquad+2\varepsilon_M(\gamma P_L+P_N)P_M^2]\\&\gamma_M\gamma_N\gamma_L[\{(\gamma P_M)(P_L^2+2P_M^2+2P_N^2)-\gamma P_L(2P_L^2+P_M^2+2P_N^2)\}(\delta_N-1)\\&\qquad+2\varepsilon_N(\gamma P_M-P_L)P_N^2]\\&\gamma_N\gamma_L\gamma_M[-(\gamma P_L)(2P_L^2+P_M^2+2P_N^2)-\gamma P_M(2P_L^2+2P_M^2+P_N^2)](\delta_L-1)\\&\qquad+2\varepsilon_L(\gamma-P_N-P_M)P_L^2\end{aligned}\right. \qquad (2.21)$$

$$+0\ (\mu^4/x^4)]$$

Here ε_L etc. are defined by

$$\gamma P_L\gamma_L=\varepsilon_L\gamma_L\gamma P_L$$

then

$\varepsilon_L=1$ for U_L of scalar of pseudoscalar meson,

$\varepsilon_L=-1$ for U_L of vector of pseudovector meson.

Since (2.20) and (2.21) are symmetric with respect to P_L, P_M and $-P_N$ and L, M and N, we have only to calculate about any one suite of π^+, π^0 and τ^+ corresponding to U_L, U_M and U_N and change cyclically P_π, P_{π^0} and $-P_\tau$ lately. Traces must be evaluated for respective sets of mesons.

§ 4. Results.

After such a laborious calculation we get at length the results. But there remain yet the following problems to be solved.

1. *Divergency.* When there are even numbers of γ_i among γ_L, γ_M and γ_N the diverging part arises from according to (2.20)

$$S_p(\gamma_L\gamma_M\gamma_N)\,(\delta_L+\delta_M+\delta_N),$$

in which convergent sets are (see (2.18))

$$t\ t\ t,\ t\ s\ ps,\ v\ v\ s,\ v\ t\ pv,\ pv\ pv\ ps \tag{2.22}$$

because of $\delta_L+\delta_M+\delta_N=0$, and

$$v\ s\ pv,\ t\ s\ s,\ t\ s\ ps,\ t\ ps\ ps,\ v\ v\ ps,\ pv\ pv\ ps, \tag{2.23}$$

because of $S_p(\gamma_L\gamma_M\gamma_N)=0$. Other cases are divergent. The infinity in these cases is let to finite or zero by regulator. The values of life times grounded by such prescription will give upper limits when future theory will be established.

When there are odd numbers of γ_i among γ_L, γ_M and γ_N, that convergency takes place in the sets

$$v\ s\ ps,\ s\ s\ pv,\ ps\ ps\ pv,\ pv\ pv\ pv,\ v\ v\ pv \tag{2.24}$$

because the trace of a divergent term is zero. Other divergent terms are brought about convergent by the regulators $\int dx\rho(x)=0$ and $\int dx\ log\ |x|\ \rho(x)=0$. Their life times obtained give upper limits, too.

2. *Ambiguity.* Among convergent cases, (2.22) and (2.23), there remains still ambiguity, except for the cases eliminated by trace, though it is dropped by regulator. But this prescription, as mentioned above, is not consistent with equivalence and divergence theorems. If such a terms is dropped by regulator, the equivalence breaks down though it should hold. On the other hand, if this term is maintained, there arises the contradiction with the divergence theorem and the gauge invariance (see in III) like as in the cases of γ-decay. Then we are puzzled which is the consistent prescription. Here we get the upper limits of life times tentatively by making use of regulator, though questions are left unsolved in future investigations.

3. *Table of forbidden sets.* Summarizing the above results, we can exhausts the whole sets except the Furry's forbidden cases, for two particles decay of a τ-meson, as shown in Table I.

The left half of the Table shows the cases of neutral theory and the right half is of symmetrical theory. The sets in upper half contain the even number of γ and are applied the regulators $\int dx\, x\, \rho(x)=0$ and $\int dx\, x\, \rho(x)\ log\ |x|=0$ and the sets in lower half contain the odd number of γ and are applied the regulators $\int dx\rho(x)=0$ and $\int dx\ log\ |x|\ \rho(x)=0$. Other notations are explained below the Table.

We show some examples of ambiguity.

Example 1. $\tau^+(S, s)$ and π^+, π^0 (V, v). Divergence theorem.

The matrix element M for this example is

Table I. Forbidden sets.

	Neutral theory						Symmetrical theory					
	2A+B			3B			3A			A+2B		
Even γ	v v s	a	D	s s s			v v t			v s pv	/	D E
	v v ps	/	D	s pv pv			t t t	a		v ps pv		
	v t pv	a	D	s s ps	/					t s s	/	
	t t s			s ps ps						t s ps		
	t t ps			pv pv ps	/	E				t pv pv		
				ps ps ps	/	*				t ps ps		
Odd γ	v v pv	/	D E	s s pv	/	* E	v v v			v s s		G
	v t s		D	s ps pv			t t t			v s ps	/	* D
	v t ps		D	ps ps pv	/	*				v pv pv		
	t t pv			pv pv pv	a	E				v ps ps		G
										t s pv		E
										t pv ps		E

Note:

A: Couplings, v and t.
B: Couplings, s pv and ps.
/: Convergent and no ambiguity.
a: Convergent but ambiguous.
* Forbidden because of no invariant matrix.
D: Forbidden when v is the vector coupling of scalar meson (Divergence theorem).
E: Equivalence theorem holds when pv is the pv coupling of ps meson.
G: Forbidden because of no gauge invariant matrix in the case of $\tau \to \pi + \gamma$ decay.
Unspecified: Divergent unless regulator is applied.

$$M = \{ -(V_i^+ V_i^0) + \frac{1}{3\varkappa^2}(V_{ij}^+ V_{ij}^0) + \cdots \} \phi_\tau$$

$$V_{ij} = \partial_i V_j - \partial_j V_i = i(P_i V_j - P_j V_i) \qquad (2.24)$$

where $V_i^+ \cdot V_i^0$ and ϕ_τ represent the wave functions of π^+, π^0 and τ mesons, and P_i is the propagation vector.

If π^+ and π^0 mesons are scalar mesons with vector couplings, we have only to do the substitution

$$V_i^+ \to P_i^+ \qquad V_i^0 \to P_i^0$$

in (2.24). But the first term does not vanish in contradiction with the divergence theorem, while the second and the subsequent terms vanish.

Example 2. $\tau^+(P_s)$ and π^+, $\pi^0(P_v, pv)$. Equivalence theorem.

In this case there should be the equivalence between pseudovector and pseudoscalar couplings of the τ^+-meson. The matrix element for pseudovector and pseudoscalar coupling is

$$M_{pv} = \text{ambiguity} + \frac{i}{30x^2} \begin{pmatrix} p^+ \\ V^+ \\ p^0 \\ V^0 \end{pmatrix} \frac{(\mu_\tau^2 - \mu_{\pi+}^2 - \mu_{\pi0}^2)}{\mu_\tau^2} \phi_2 + \cdots \tag{2.25}$$

where μ_π^+ and μ_π^0 are the masses of π^+ and π^0 or *vice versa*, while for the case of pseudocalar coupling

$$M_{ps} = \left(\frac{i}{3x} + \frac{i}{60x^3} \frac{\mu_\pi^2 \cdot - \mu_\pi^2 + - \mu_{\tau 0}^2}{\mu_\tau^2} + \cdots \right) \begin{pmatrix} p^+ \\ V^+ \\ p^0 \\ V^0 \end{pmatrix} \phi_\tau . \tag{2.26}$$

The equivalence would be hold if $M_{pv} = 2xM_{ps}$. But the first term in the right hand side of (2.25) has ambiguity and should vanish if the prescription of Schwinger could be used. Then the equivalence does not hold against the general theorem. This difficulty is not solved by the ordinary regulator condition. If one forces to make the equivalence hold, one may add the new condition of the regulator $\int dx\, \rho(x)/x = 0$ as suggested by Ozaki in relation with the γ-decay of a neutral pseudoscalar meson. We suppose, however, that such a new condition may be too severe and, for example, results in vanishing the anomalous magnetic moment of an electron. Therefore, this contradiction will be left unsolved. Between the second and subsequent terms the equivalence, of course, holds.

Example 3. π^+, π^0 and $\tau + (P_v, t)$.

The matrix element for this case consists in only one term

$$M = -S_p \{\gamma_{ij}\gamma_5\gamma_{kl}\gamma_5\gamma_{mn}\gamma_5\} \cdot U_{ij}^{\pi+} U_{kl}^{\pi 0} U_{mn}^{\tau+} \frac{x}{\mu_\pi \mu_{\pi 0} \mu_\tau} . \tag{2.27}$$

This term is convergent and gives the life time $\sim 10^{-19}$ sec., wheras it vanishes by the regulator conditions $\int dx\, \rho(x) = 0$ and gives the infinite life time. Which of both prescriptions is correct will be solved by the future investigation.

4. *Mass dependence.* The dependence of the life on the masses of nucleon and τ-meson. x and μ_τ, is $(x/\mu_\tau)^2$ or $(x/\mu_\tau)^4$ according as the number of γ_i in γ_L, γ_M and γ_N is even or odd, provided that the regulator is applied in the case of even number.

Nevertheless, there are special sets in the relation of masses of mesons. The first example is the set composed of $\tau(S, v)$ and π^+ and π^0 with the same kind of types and couplings in symmetrical theory. In this set the marix element vanishes if the masses of π^+ and π^0 mesons are equal in each other.

The second example is the set $\tau(Ss)$ and π^+, π^0 (P, ps). In this set the matrix element vanishes if $\mu_\tau = \sqrt{6}\ \mu_\pi$. It may be a noticeable fact that the mass of a τ-meson, 700 m, provided $\mu_\pi = 286$ m, is similar to the mass value

Table II. Life times of $\tau \to \pi + \pi^0$ decay.

τ: Scalar meson.

Coupling of τ		Scalar				Vector			
Coupling of π, π^0		$f f_0$	$f g_0$	$g f_0$	$g g_0$	$f f_0$	$f g_0$	$g f_0$	$g g_0$
s, s	n	8.0×10^{-17} R_1	×	×	D	×	D	D	×
	s	×	1.4×10^{-15} R_0	1.4×10^{-15} R_0	×	Δ	Δ	×	D
s, v	n	×	×	D	D	D	D	×	×
	s	1.5×10^{-15} R_0	1.9×10^{-15}	×	×	×	×	3.9×10^{-12} R_0	1.8×10^{-14} R_1
s, pv	n	*	*	*	*	*	*	*	*
	s	*	*	*	*	*	*	*	*
s, ps	n	*	*	*	*	*	*	*	*
	s	*	*	*	*	*	*	*	*
v, v	n	2.2×10^{-16} a	5.0×10^{-15} R_0	5.0×10^{-15} R_0	2.1×10^{-17} R_1	×	×	×	×
	s	×	×	×	×	Δ	3.3×10^{-15} R_1	3.3×10^{-15} R_1	Δ
v, pv	n	×	4.6×10^{-15} R_0	×	3.2×10^{-17} R_1	D	×	D	×
	s	1.0×10^{-15}	×	2.1×10^{-15} R_0	×	×	9.0×10^{-15} R_1	×	2.0×10^{-12} R_0
v, ps	n	*	*	*	*	*	*	*	*
	s	*	*	*	*	*	*	*	*
pv, pv	n	1.3×10^{-17} R_1	×	×	2.8×10^{-17} R_1	×	D	D	×
	s	×	2.7×10^{-15} R_0	2.7×10^{-15} R_0	×	Δ	×	×	Δ
pv, ps	n	2.3×10^{-15} R_0	1.6×10^{-17} R_1	×	×	×	×	D	D
	s	×	×	2.0×10^{-15}	6.0×10^{-14}	1.8×10^{-14} R_1	4.0×10^{-13} R_0	×	×
ps, ps	n	9.7×10^{-15} R_1	1.8×10^{-10} R_0	1.8×10^{-15} R_0	1.3×10^{-17} R_1	×	×	×	×
	s	×	×	×	×	Δ	1.0×10^{-14} R_1	1.0×10^{-14} R_1	Δ

τ: Vector meson.

Coupling of τ		Vector				Tensor			
Coupling of π, π^0		$f f_0$	$f g_0$	$g f_0$	$g g_0$	$f f_0$	$f g_0$	$g f_0$	$g g_0$
s, s	n	×	D	D	×	×	D	D	×
	s	2.7×10^{-14} R_0	×	×	1.8×10^{-15} R_0	6.0×10^{-15}	×	×	7.0×10^{-17} R_1
s, v	n	3.3×10^{-16} a	3.7×10^{-14} R_0	×	×	4.7×10^{-14} a	2.9×10^{-15} R_1	×	×
	s	×	×	5.0×10^{-16} R_0	1.7×10^{-17} R_1	×	×	2.3×10^{-17} R_1	7.0×10^{-16} R_0
s, pv	n	×	1.4×10^{-14} R_0	D	×	×	9.0×10^{-16}	D	×
	s	2.9×10^{-15}	×	×	3.6×10^{-17} R_1	3.0×10^{-14} R_0	×	×	1.6×10^{-15} R_0

s, ps	n s	* *	* *	* *	* *	* *	* *	* *	* *
v, v	n s	× 3.7×10^{-16} R_0	× 1.2×10^{-17} R_1	× 1.2×10^{-17} R_1	× 1.6×10^{-10} R_0	× 1.5×10^{-17} R_0	× 5.4×10^{-17} R_0	× 5.4×10^{-17} R_0	× 2.6×10^{-17} a
v, pv	n s	2.7×10^{-15} a ×	× 1.3×10^{-17} R_1	2.0×10^{-16} a ×	× 1.2×10^{-13} R_0	4.3×10^{-17} a ×	× 6.0×10^{-16} R_0	1.3×10^{-14} R_0 ×	× 4.4×10^{-17} a
v, ps	n s	3.2×10^{-16} ×	3.5×10^{-15} a ×	1.1×10^{-15} R_0 ×	5.0×10^{-17} a ×	1.0×10^{-13} R_0 ×	1.1×10^{-16} a ×	6.0×10^{-16} ×	3.7×10^{-15} R_0 ×
pv, pv	n s	× 2.0×10^{-15} R_0	4.1×10^{-17} a ×	4.1×10^{-17} a ×	× 1.6×10^{-15} R_0	× 3.0×10^{-16} R_1	4.5×10^{-14} R_0 ×	4.5×10^{-14} R_0 ×	× 2.3×10^{-16} a
pv, ps	n s	× 6.0×10^{-17} R_1	× 1.4×10^{-14} R_1	1.3×10^{-15} R_0 ×	8.0×10^{-17} a ×	× 3.0×10^{-15} R_0	× 5.6×10^{-15} R_1	4.4×10^{-17} R_1 ×	8.2×10^{-15} R_0 ×
ps, ps	n s	× 1.6×10^{-14} R_0	× 1.2×10^{-16} R_1	× 1.2×10^{-16} R_1	× 1.1×10^{-15} R_0	× 7.0×10^{-16}	× 7.0×10^{-16}	× 7.0×10^{-15} a	× 1.5×10^{-15} R_1

τ: Pseudovector meson.

Coupling of τ		Pseudovector				Tensor			
Coupling of π, π^0		$f\,f_0$	$f\,g_0$	$g\,f_0$	$g\,g_0$	$f\,f_0$	$f\,g_0$	$g\,f_0$	$g\,g_0$
s, s	n s	* *	* *	* *	* *	* *	* *	* *	* *
s, v	n s	× 2.9×10^{-15}	× 6.0×10^{-15} R_0	D ×	D ×	1.2×10^{-13} R_0 ×	9.0×10^{-16} R_1 ×	× 2.6×10^{-14} R_1	× 1.3×10^{-14} R_0
s, pv	n s	1.3×10^{-16} R_1 ×	× 1.2×10^{-15} R_0	× 2.2×10^{-15} R_0	D ×	× 4.2×10^{-14} R_0	4.0×10^{-16} R_1 ×	D ×	× 7.0×10^{-15} R_0
s, ps	n s	6.3×10^{-14} R_0 ×	4.4×10^{-16} R_1 ×	× 5.0×10^{-14} R_1	× 6.0×10^{-16}	× 6.0×10^{-14}	× 1.6×10^{-13}	D ×	D ×
v, v	n s	3.8×10^{-13} a ×	3.4×10^{-16} a ×	3.4×10^{-16} a ×	4.2×10^{-12} R_0 ×	× 4.8×10^{-15} R_1	× 5.0×10^{-15} R_0	× 5.0×10^{-15} R_0	× 3.5×10^{-16} a
v, pv	n s	× 1.0×10^{-15} R_0	6.6×10^{-16} a ×	× 5.4×10^{-17} R_1	1.6×10^{-14} R_0 ×	1.1×10^{-12} a ×	× 3.6×10^{-15} R_0	4.6×10^{-15} R_0 ×	× 1.1×10^{-15} a
v, ps	n s	× 1.0×10^{-14} R_1	× 1.1×10^{-14} R_1	× 3.2×10^{-15} R_0	× 3.6×10^{-14} R_1	6.5×10^{-14} R_0 ×	7.0×10^{-16} a ×	1.0×10^{-14} R^1 ×	2.8×10^{-13} R_0 ×
pv, pv	n s	1.5×10^{-13} ×	× 7.0×10^{-17} R_1	× 7.0×10^{-17} R_1	4.4×10^{-14} R_0 ×	× 2.0×10^{-16} R_1	1.1×10^{-14} R_0 ×	1.1×10^{-14} R_0 ×	× $\infty\,n$

pv, ps	n	2.9×10^{-15}	6.2×10^{-14} a	×	×	×	×	2.6×10^{-14} R_1	1.3×10^{-14} R_0
	s	×	×	1.4×10^{-14} R_0	1.4×10^{-16} R_1	1.2×10^{-13} R_0	1.3×10^{-15} R_1	×	×
ps, ps	n	*	*	*	*	*	*	*	*
	s	*	*	*	*	*	*	*	*

τ: Pseudoscalar meson.

Coupling of τ		Pseudoscalar				Pseudovector			
Coupling of π, π^0		$f\,f_0$	$f\,g_0$	$g\,f_0$	$g\,g_0$	$f\,f_0$	$f\,g_0$	$g\,f_0$	$g\,g_0$
s, s	n	*	*	*	*	*	*	*	*
	s	*	*	*	*	*	*	*	*
s, v	n	*	*	*	*	*	*	*	*
	s	*	*	*	*	*	*	*	*
s, pv	n	2.3×10^{-15} R_0	×	×	D	1.5×10^{-16} R_1	×	×	D
	s	×	2.0×10^{-15}	2.4×10^{-17} R_1	×	×	1.6×10^{-12}	6.0×10^{-16} R_0	×
s, ps	n	1.0×10^{-16} R_1	3.4×10^{-15} R_0	×	×	1.4×10^{-16} R_0	2.0×10^{-16} R_1	×	×
	s	×	×	4.9×10^{-16} R_0	1.4×10^{-17} R_1	×	×	3.4×10^{-17} R_1	5.0×10^{-15} R_0
v, v	n	1.1×10^{-16}	3.8×10^{-15} R_0	3.8×10^{-15} R_0	6.0×10^{-18} R_1	1.0×10^{-14} a	3.7×10^{-17} a	3.7×10^{-17} a	5.5×10^{-16} R_0
	s	×	×	×	×	×	×	×	×
v, pv	n	×	4.1×10^{-15} R_0	×	6.6×10^{-18} R_1	×	3.5×10^{-17} a	×	6.4×10^{-15} a
	s	1.1×10^{-17} R_1	×	3.2×10^{-15} R_0	×	1.0×10^{-15} R_0	×	2.0×10^{-16} R_1	×
v, ps	n	×	×	×	×	×	×	×	×
	s	6.0×10^{-16} R_0	1.2×10^{-17} R_1	2.2×10^{-16}	2.3×10^{-15} a	4.0×10^{-17}	1.1×10^{-15} R_0	2.2×10^{-16} R_1	1.8×10^{-16} R_1
pv, pv	n	1.0×10^{-15}	×	×	1.2×10^{-17} R_1	3.8×10^{-14} a	×	×	1.2×10^{-15}
	s	×	4.6×10^{-15} R_0	4.6×10^{-15} R_0	×	×	7.0×10^{-16} R_1	7.0×10^{-16} R_1	×
pv, ps	n	*	*	*	*	*	*	*	*
	s	*	*	*	*	*	*	*	*
ps, ps	n	*	*	*	*	*	*	*	*
	s	*	*	*	*	*	*	*	*

Note:

× : Forbidden by Furry's theorem.

* : Forbidden because of no Lorentz invariant matrix.

Δ: Forbidden by $\mu^{\pi\pm}=\mu^{\pi^0}$.

D: Forbidden when v is the vector coupling of scalar meson (Divergence theorem).

∞a: Forbidden when τ, π and π^0 are the tensor couplings of pseudovector mesons by making use of regulator.

a: Convergent but ambiguous, applying regulator.

R_0: Divergent but dropped by regulator $\int dx\,\rho(x)=0$, $\int dx\,log|x|\,\rho(x)=0$,

R_1: Divergent but dropped by regulator $\int dx\,x\rho(x)=0$, $\int dx\,x\,log|x|\,\rho(x)=0$.

measured by Wagner and Cooper,[32] 725 ± 40 m. Such a kind of τ-meson would have long life, if the pseudovector coupling of pseudoscalar meson were not effective.

5. *Table of life times.* The life times for $\tau \rightarrow \pi + \pi^0$ decay are represented in Table II for all possible combinations. To calculate the life times we assume for simplicity the values of masses and coupling constants as follows.

$$\mu_\tau = 900 \text{ m}, \qquad \mu_{\pi\pm} = \mu_{\pi 0} = 300 \text{ m}, \tag{2.27}$$

$$f^2 = g^2 = 10^{-1}, \quad F^2 = G^2 = 10^{-3}. \tag{2.28*}$$

f and g are the coupling constants of $\pi^\pm$ and π^0 mesons with a nucleon, corresponding to the couplings without and with differentiation, respectively. F and G are corresponding quantities for τ-meson. These values adopted are only conventional ones and will be discussed lately. In this Table $\pi^\pm$- an π^0-mesons are symmetrical and can be interchanged in each other. *(to be continued)*

42) N. Wagner and D. Cooper, Phys. Rev. **76** (1949), 449.

* Concerning to the magnitude of coupling constants we ignore the equivalent relations. f^2, g^2 etc. should be expressed by $f^2/4\pi$, $g^2/4\pi$ etc. in Heaviside unit, but 4π is dropped here for simplicity.

Letters to the Editor

On the Decay of a Heavy Meson into Lighter Mesons.

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January 7, 1950

Existence of a heavy meson (τ-meson) having the mass about 700~1000 times of an electron and the fact it disintegrates into π-mesons is almost confirmed by recent experiments. For example, C. D. Rochester and C. C. Butler[1] have reported the processes which are considered to be $\tau^\pm \to \pi^\pm + \pi^0$ and $\tau^0 \to \pi^+ + \pi^-$ and C. F. Powell and others[2] have observed the decay process in which a τ-meson disintegrated into three π-mesons. Furthermore, L. Leprince-Ringuet[3] obtained the interesting photograph in which a τ-meson gave rise to star and a σ-meson, produced from that star, led to the next star. Among them, Powell and his coworker's experiment seems to be most convincing and Leprince-Ringuet's photograph indicates us the fact that the coupling of τ-meson with nucleons is not so small because of the production of star. From the evidence mentioned above and the unstability of a τ-meson, it is reasonable to consider that the process occurs via nucleon field and a τ-meson is a Boson. If a τ-mesón is a Fermion, Powell's process occurs as the following:

1) $\tau^+ \to N + P^+ + \tau^0 \to \pi^+ + P^- + P^+ + \tau^0$
$\to \pi^+ + \pi^+ + P^- + N + \tau^0$
$\to \pi^+ + \pi^+ + \pi^- + \tau^0,$ (τ_0 spin $\frac{1}{2}$)

2) $\tau^+ \to P + P^- + \tau^{1+} \to \pi^+ + N + P^- + \tau^{1+}$
$\to \pi^+ + \pi^- + \tau^{1+}$ (τ^{1+} spin $\frac{1}{2}$)

As for the case 1), the momentum law being satisfied by three particles (Powell's evidence), it is difficult to consider the other uncharged particle besides charged π mesons and the mass of a τ meson may be not so heavy that it disintegrates 4 particles. On the other hand, in the process 2), τ^{1+} meson is not π, because it is a Fermion. Therefore in this case, we cannot help considering the new particle which has a almost same mass as that of a π-meson and spin $\frac{1}{2}$ and whose interaction with nucleon is fairly strong. Judging from this point of view, it seems to be reasonable to consider a τ meson as a Boson.

If a τ meson is a Boson, Powell's process occurs as the fourth order one in the perturbation method; $\tau^+ \to P + N \to \pi^+ + N + N \to \pi^+ + \pi^- + P + N \to \pi^+ + \pi^- + \pi^+$, while the processes $\tau^\pm \to \pi^\pm + \pi^0$ and $\tau^\pm \to \pi^\pm + \gamma$ occur generally more rapidly than the former, because of the third order process. Therefore, in order to explain Powell's experimental result, we ought to seek the forbidden case of $\tau^\pm \to \pi^\pm + \pi^0$, and $\tau^\pm \to \pi^\pm + \gamma$. In the process $\tau^\pm \to \pi^\pm + \gamma$, the following cases are forbidden[4]

1) when both τ and π meson have spin 0.
2) when τ or π meson is the scalar meson with vector coupling.

In the process $\tau^\pm \to \pi^\pm + \pi^0$, we obtained the following selection rules[5]. Namely, if we denote wave functions of τ and π mesons by U_1, V_2, W_3 and Dirac's matrices of their interaction with nucleon by O_1, O_2, O_3 respectively,

$$\left.\begin{array}{lr} Sp(O_1O_2O_3+\varepsilon O_3O_2O_1) & \\ \qquad U_1V_2W_3=0 & (1) \\ Sp(O_1O_2\gamma_\alpha O_3-\varepsilon O_3\gamma_\alpha O_2O_1) & \\ \qquad U_1\frac{\partial}{\partial X_\alpha}(V_2W_3)=0 & (2) \end{array}\right\} \text{forbidden}$$

and specially in the case where (1) vanishes identically owing to γ_5, then

$$Sp(O_1\gamma_\alpha O_2\gamma_\beta O_3+\varepsilon O_3\gamma_\beta O_2\gamma_\alpha O_1)$$
$$U_1\frac{\partial^2}{\partial X_\alpha \partial X_\beta}(V_2W_3)=0 \quad (3) \qquad \text{forbidden}$$

where ε equals to 1 for symmetrical theory and -1 for neutral one. From the rules mentioned above, the cases of pseudoscalar τ meson and scalar and pseudoscalar π mesons ($ps \to s+s$, $ps \to ps+ps$) are forbidden for both symmetrical and neutral theory. From these results following choices for $\tau^+ \to \pi^+ + \pi^- + \pi^+$ are allowed, that is:

1) τ is pseudoscalar and π is pseudoscalar.
2) τ is pseudoscalar and π is scalar.

And also the analogous selection rules are obtained for the fourth order transition process, that is, if we denote the wave functions of τ and π mesons by Y_4, U_1, V_2, W_3 and their interaction matrices by O_4, O_1, O_2, O_3 respectively,

$$\left.\begin{array}{r} Sp(O_1 O_2 O_3 O_4 + O_4 O_3 O_2 O_1) \\ Y_4 U_1 V_2 W_3 = 0 \\ Sp(\gamma_\alpha O_1 O_2 O_3 O_4 - O_4 O_3 O_2 O_1 \gamma_\alpha) \\ Y_4 \dfrac{\partial}{\partial X_\alpha}(U_1 V_2 W_3) = 0 \end{array}\right\} \text{forbidden.}$$

Of course, the same anxieties as in (1) are required when the form $Sp(\gamma_5 \gamma_\alpha \gamma_\beta)$ appears. Applying the rules to these two cases, we find that the all combination of the former 1) are allowable and those of the latter 2) vanish. Therefore we have to calculate only the case where both τ and π are of pseudoscalar. The decay life times for possible 4 types are calculated.

1) $G_1 g_1^3$ (both τ and π pseudoscalar interactions)

$$\tau_0^{-1} = \frac{(G_1 g_1^3)^2 \varkappa_\tau c}{400\pi^3 (\hbar c)^4} \left(\frac{\varkappa_\tau}{\varkappa}\right)^4 \int_a^{(1-3a^2)/2} dM \int_{(1-M-A)/2}^{(1-M+A)/2} dN$$
$$\{79a^2 - 41 + (44M + 58N)\}^2,$$

$\tau_0 = 1{,}5 \times 10^{-10}$ sec.

2) $G_2 g_1^3$ (τ pseudovector, π pseudoscalar interactions)

$$\tau_0^{-1} = \frac{9(G_2 g_1^3)^2 \varkappa_\tau c}{144\pi^3 (\hbar c)^4} \left(\frac{\varkappa_\tau}{\varkappa}\right)^2 \int_a^{(1-3a^2)/2} dM \int_{(1-M-A)/2}^{(1-M+A)/2} dN$$
$$(2M + 3N - 1)^2,$$

$\tau_0 = 1.8 \times 10^{-12}$ sec.

3) $G_1 g_2^3$ (τ pseudoscalar, π pseudovector interactions)

$$\tau_0^{-1} = \frac{9(G_1 g_2^3)^2 \varkappa_\tau c}{144\pi^3 (\hbar c)^4} \left(\frac{\varkappa_\tau}{\varkappa}\right)^2 \left(\frac{\varkappa_\tau}{\varkappa_\pi}\right)^6 \int_a^{(1-3a^2)/2} dM \,.$$
$$\int_{(1-M-A)/2}^{(1-M+A)/2} dN \,\{M^2 + 10MN + 4N^2 - (8 + 19a^2)M - \left(\frac{19}{2} + a^2\right)N + 5 + \frac{5}{3}a^2 - \frac{a^4}{4}\}^2,$$

$\tau_0 = 1 \times 10^{-14}$ sec.

4) $G_2 g_2^3$ (both τ and π pseudovector interactions)

$\tau_0 = 7.7 \times 10^{-16}$ sec.

where we used the following values and abbrevations:

$$A = \sqrt{\frac{(M^2 - a^2)(1 - 3a^2 - 2M)}{1 + a^2 - 2M}}, \quad \varkappa_\tau = M_\tau c/\hbar,$$
$$\varkappa_\pi = M_\pi c/\hbar, \quad \varkappa = M_N c/\hbar, \quad a = \varkappa_\pi/\varkappa_\tau,$$
$$g^2/\hbar c = 1/10, \quad G^2/\hbar c = 1/10^3, \quad M_\tau = 1000\,\text{m},$$
$$M_\pi = 300\,\text{m}, \quad M_N = 1840\,\text{m},$$

and the diverging and ambiguous integrals appeared in the calculations are dropped off by regulators $\int \rho(\varkappa) \log|\varkappa| d\varkappa = 0$, $\int \rho(\varkappa) d\varkappa = 0$ and $\int \rho(\varkappa) \varkappa d\varkappa = 0$. In these cases 3) and 4) are too short to be considered as reasonable values and can not explain Powell's experiments. Therefore, types of interactions of π mesons seem to be pseudoscalar. Besides the explanations mentioned above, there is other possibilities, that is, the processes $\tau^+ \to \pi^+ + \tau^0$ and $\tau^0 \to \pi^+ + \pi^0$ occur successively. But as was seen in part II,[5] its life time is too short to explain Powell's experiments and their possibilities are still undetermined unless the exact values of masses of τ and τ^0 mesons and their coupling constants with nucleons are obtained. After finishing of these calculations, we obtained S. Power's result[7] concerning the same problem which seems too long compared with ours. Details will soon appear in Prog. Theor. Phys.

1) C. D. Rochester and C. C. Butler; Nature, **160** (1947), 755.
2) R. Brown, U. Camerini, P. H. Fowler, H. Muirhead, C. F. Powell and D. M. Riston; Nature, **163** (1949), 417.
3) L. Leprince-Ringuet; Rev. Mod. Phys., **21** (1949), 42.
4) S. Ôneda, S. Sasaki and S. Ozaki, III; Prog. Theor. Phys., **5** (1950), 165.
5) S. Sasaki, S. Ôneda and S. Ozaki, **II** ibid, **5**

(1950), 25.
6) S. Ozaki, S. Ôneda and S. Sasaki, I, ibid, **4** (1949), 524.
S. Sasaki, S. Ôneda and S. Ozaki, IV, ibid. **5** (1950), in press
7) S. Power; Phys. Rev., **76** (1949), 865.

Effect of Nuclear Motion on the Fine Structure of Hydrogen.

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January 16, 1950.

The relativistic 16-component wave equation including the interaction $(e^2/2r)[(\boldsymbol{\alpha}_e\boldsymbol{\alpha}_M)+(\boldsymbol{\alpha}_e\boldsymbol{r})(\boldsymbol{\alpha}_M\boldsymbol{r})/r^2]$ was applied for the system of a proton and an electron. By eliminating its angular part for a given total angular momentum F, M, two sets of simultaneous equations were obtained for eight radial functions respectively. Each set of the equations is reducible, in the special case $F=0$, to a set of equations for two radial functions; each set corresponds to the state $P_{\frac{1}{2}}$ or $S_{\frac{1}{2}}$ respectively. The equations were solved by the usual perturbation method in power series in $\beta=m/M$ and the energy values were obtained, in each power of β, further in each power of α^2 $(\alpha=e^2/c\hbar)$. The results have shown: The term of order $\beta\alpha^2$ in unit of mc^2 gives simply the reduced mass correction in accordance with those of Breit and Brown[1] and of other former works. Of the term of order $\beta\alpha^4$ the only on n dependent part also agrees with the result of Breit and Brown, but differs slightly from the simple reduced mass correction, i. e. by a term proportional to $1/n^4$; the remaining part corresponds exactly to the hyperfine splitting for each of the two states $n^2P_{\frac{1}{2}}$, $F=0$ and $n^2S_{\frac{1}{2}}$, $F=0$ of an idealized proton having one nuclear magneton. The terms of order $\beta^2\alpha^2$, and of order α^2 in remaining power of β give, however, twice as much as the reduced mass corrections. Thus we may conclude that within terms of order $\beta\alpha^4$ as well as of order α^2 in every power of β no displacement occurs between the levels $n^2P_{1/2}$ and $n^2S_{1/2}$.

1) G. Breit and G. E. Brown, Phys. Rev. **74**, 1278, (1948).

On the Electromagnetic Properties of Mesons.

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The Quantum Electrodynamics recently developed by Tomonaga and Schwinger was able to explain the behavior of an electron in the electromagnetic field quite well. But there remain many questions unsolved in applying this theory to the problems concerning meson. With an intention to make clearer the present situation of the meson theory, we have calculated the electromagnetic correction to the mesonic charge-current which involves quantities observable in principle. Relativistic extension of Duffin-Kemmer formalism is used to follow the formal analogy of the electron as far as possible. It is seen that the result can be identified with that of the usual Proca and Klein-Gordon formalisms.

Our interaction Hamiltonian is

$$H_x[\sigma]=-ie\psi^\dagger\beta_\mu\psi A_\mu+\frac{e^2}{m}\psi^\dagger\beta_\mu (1+(\beta n)^2)\beta_\nu\psi A_\mu A_\nu \qquad (1)$$

where ψ is a free particle operator satisfying

$$(\beta_\lambda\partial_\lambda+m)\psi=0,\ (\lambda=1,2,3,4\,;\ x_4=it\,;\ \hbar=c=1), \qquad (2)$$

and

$$\psi^\dagger=\psi^*(2\beta_4{}^2-1). \qquad (3)$$

n_ν is an unit vector normal to the surface. The commutation relations are

$$[\psi_\rho(x),\ \psi^\dagger_\sigma(x')]=\frac{1}{i}B_{\rho\sigma}D(x-x'),$$

$$B\equiv\beta\partial-m+\frac{1}{m}\square^2-\frac{1}{m}(\beta\partial)^2 \text{ [1]}$$

$$[A_\mu(x),\ A_\nu(x')]=i\delta_{\mu\nu}D_0(x-x'),$$
$$\text{others are zero,} \qquad (4)$$

where $D(x)$ and $D_0(x)$ are D-functions of meson and photon respectively. The Schroedinger equation corresponding to (1) can be solved in the form of the S-matrix

$$S=\sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int_{-\infty}^{\infty}(dx_1)\cdots(dx_n)P(H_{x_1}[\sigma_1],$$
$$H_{x_2}[\sigma_2],\ \cdots,\ H_{x_n}[\sigma_n]) \qquad (5)$$

in which $P(\quad)$ is a chronological ordering operator of Dyson. This comes out to be independent of the choice of families of intermediate surfaces in spite of the explicit dependence of each $H_{x_i}[\sigma_i]$ on the normal to the surface, which is the content of the integrability of (1).[2] The lowest order correction to the mesonic current is found when we extract from this S-matrix the second order correction to the elastic scattering of a meson scattered only once by an external electromagnetic potential. It is found that

$$(\delta j_\mu^{(2)}(x))_{1,0}=\frac{i}{2}\int_{-\infty}^{\infty}(dx')(dx'')\varepsilon(x-x')[j_\mu(x),\ j_\nu(x')]_0\bar{D}_0(x'-x'')j_\nu(x'')$$
$$-\frac{i}{2}e^3\int_{-\infty}^{\infty}(dx')(dx'')\psi^\dagger(x')K_\mu(x'-x,\ x-x'')\psi(x'')$$
$$+\frac{i}{2}e\int_{-\infty}^{\infty}(dx')(dx'')[\psi^\dagger(x)\beta_\mu B\bar{D}(x-x')\phi(x')+\phi^\dagger(x')B\bar{D}(x'-x)\beta_\mu\psi(x)]$$
$$+\frac{i}{2}\int_{-\infty}^{\infty}(dx')\varepsilon(x-x')[j_\mu(x),\ H^{\rm self}(x')] \qquad (6)$$

where

$$K_\mu(\xi,\eta)=\beta_\nu(\bar{D}_0(\xi+\eta)B\bar{D}(\xi)\beta_\mu BD^{(1)}(\eta)+\bar{D}_0(\xi+\eta)BD^{(1)}(\xi)\beta_\mu B\bar{D}(\eta)$$
$$+D_0^{(1)}(\xi+\eta)B\bar{D}(\xi)\beta_\mu B\bar{D}(\eta))\beta_\nu \qquad (7)$$

and

$$\phi(x)=-\frac{e^2}{2}\int_{-\infty}^{\infty}(dx')\beta_\nu(\bar{D}_0(x-x')BD^{(1)}(x-x')+D_0^{(1)}(x-x')B\bar{D}(x-x'))\beta_\nu\psi(x'). \qquad (8)$$

(We have used $D^{(1)}(x)$ and $\bar{D}(x)$ instead of $D_F(x)$ for convenience of calculation.) The first term in (6) represents the effect of vacuum polarization which is discussed by many authors;[3] it is

$$(\delta j_\mu^{(2)}(x))_{1,0}^{\rm vac.\ pol.}=\begin{cases}-\frac{\alpha}{6\pi}\left(\log\frac{K+K_0}{m}-1\right)j_\mu(x)-\frac{\alpha}{24\pi}\frac{\Box^2}{m^2}\int_{-\infty}^{\infty}(dx')F_2(x-x')j_\mu(x') & \text{for spin 0}\\ -\frac{\alpha}{2\pi}\left(\frac{KK_0}{m^2}-\log\frac{K+K_0}{m}\right)j_\mu(x)-\frac{\alpha}{24\pi}\left(4\log\frac{K+K_0}{m}-5\right)\frac{\Box^2}{m^2}j_\mu(x) & \\ \quad -\frac{\alpha}{8\pi}\frac{\Box^2}{m^2}\left(1+\frac{\Box^2}{3m^2}\right)\int_{-\infty}^{\infty}(dx')F_2(x-x')j_\mu(x') & \text{for spin 1}\end{cases} \qquad (9)$$

where $K_0=(K^2+m^2)^{1/2}$ and

$$F_n(x)=\frac{1}{(2\pi)^4}\int(dk)e^{ikx}\int_0^1 dv\frac{v^{2n}}{1+\frac{k^2}{4m^2}(1-v^2)}.$$

$H^{\mathrm{self}}(x)$ in the last term of (6) can not really be the e^2-electromagnetic self-energy of a meson as it is, since this destroys the integrability of the theory when subtracted from (1) and thus (6) becomes dependent on the choice of the surface family. It is possible of course to modify at each stage of approximation so as to preserve the integrability. The effect of this modification is merely to drop the normal dependent part in the last term of (6) which arises when $\varepsilon(x)$ is combined with $D(x)$. $\phi(x)$ can be expanded in powers of $\beta\partial+m$; the zeroth power term gives rise to a term which is cancelled by the last term of (6) when inserted in the third term of (6), the remaining parts give terms of charge renormalization type. Half the latter is found to be just equal to the charge renormalization part in the second term of (6) with opposite sign and thus it is not necessary at all to renormalize the charge in these terms which is quite similar to the situation encountered in the case of an electron.[4] The current which arises from the remaining part of K_μ is

$$\begin{aligned}
&\frac{a}{4\pi}\left(\log\frac{K+K_0}{m}-\frac{5}{4}\right)\left[\frac{1}{12}\frac{\Box^2}{m^2}j_\mu(x)+\frac{ie}{m}\frac{\partial}{\partial x_\nu}\left\{\left(\frac{5}{2}-\frac{\eta}{3}\right)\phi^\dagger s_{\mu\nu}\phi-\phi^\dagger\beta_0{}^2 s_{\mu\nu}\phi\right\}\right]\\
&\quad-\frac{a}{4\pi}\frac{\Box^2}{m^2}\left[\frac{1}{9}j_\mu(x)+\left(\log u_0+\frac{67}{48}\right)\int_{-\infty}^{\infty}(dx')(F_0(x-x')+F_1(x-x'))j_\mu(x')\right.\\
&\qquad\left.-\left(\frac{\eta}{12}+\frac{1}{48}\frac{\Box^2}{m^2}\right)\int_{-\infty}^{\infty}(dx')F_1(x-x')j_\mu(x')\right]\\
&\quad-\frac{i\cdot ea}{4\pi m}\frac{\partial}{\partial x_\nu}\left[\left(\frac{\eta}{18}-\frac{7}{12}\right)\phi^\dagger s_{\mu\nu}\phi+\frac{7}{12}\phi^\dagger\beta_0{}^2 s_{\mu\nu}\phi+\frac{19}{24}\int_{-\infty}^{\infty}(dx')F_0(x-x')\phi'^\dagger s_{\mu\nu}\phi'\right.\\
&\left.+\left\{\frac{5}{24}-\left(\frac{5}{8}-\frac{\eta}{12}\right)\frac{\Box^2}{m^2}\right\}\int_{-\infty}^{\infty}(dx')F_1(x-x')\phi'^\dagger s_{\mu\nu}\phi'+\left(-\frac{1}{12}+\frac{\Box^2}{4m^2}\right)\int_{-\infty}^{\infty}(dx')F_1(x-x')\phi'^\dagger\beta_0{}^2 s_{\mu\nu}\phi'\right]
\end{aligned}\tag{10}$$

where $s_{\mu\nu}=\beta_\mu\beta_\nu-\beta_\nu\beta_\mu$ and $u_0\to 0$ means the infrared catastrophe. (9) with its first term removed by charge renormalization and the current (10) added together form the observable (at least in principle) part of the correction (6). This final result is logarithmically divergent and therefore is not to be compared with experiments (which is however a result expected by a general consideration from the beginning). (10) has a common form both for vector and scalar mesons which is only distinguished by the special representations of β-matrices (10–10 matrix for vector and 5–5 matrix for scalar) and further by η($=6$ and 4 for vector and scalar mesons respectively). To interpret this result literally is, however, fairly dangerous. For example, from (10) the scalar meson also seems to have a magnetic moment. In fact this is not curious since (2) should be regarded as a wave equation of a sort of particle with spin 1 even in the special representation of 5-rowed β's. In order to obtain a result which is related to the one obtained by the Klein-Gordon equation, we must go from the Duffin-Kemmer formalism to the usual tensor form by the well-known relation between the components of wave functions. In the scalar case the tensor form of

$ie\frac{\partial}{\partial x_\nu}(\psi^\dagger s_{\mu\nu}\psi)$ is easily found equal to that of $-\frac{1}{2m}\Box^2 j_\mu(x)$ and (10) may be written as

$$-\frac{a}{4\pi}(\log u_0+1)\frac{\Box^2}{m^2}\int_{-\infty}^{\infty}(dx')(F_0(x-x') + F_1(x-x'))j_\mu(x'). \qquad (11)$$

Thus there appears no magnetic moment in the Klein-Gordon formalism. In the Proca case we encounter a logarithmically divergent anomalous magnetic moment as well as an electric quadrupole moment.

Detailed calculations and discussions will be published in a later issue of this journal.

1) This commutation relation is suggested in the paper of R. P. Feynman, Phys. Rev. **76** (1949), 769.
2) P. T. Matthews, Phys. Rev. **76** (1949), 684.
3) M. Neuman and W. H. Furry, Phys. Rev. **76** (1949), 1677. R. G. Moorhouse, Phys. Rev. **76** (1949), 1691.
4) Discussions about similar circumstances are given in detail by Karplus and Kroll (Phys. Rev. in press).

Nonsingular Tensor Force in Pseudoscalar Meson Theory.

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Several authors[1] have pointed out that the so-called r^{-3}-singularity in the non-relativistic nuclear potential derived from the pseudoscalar meson theory is caused by the inadequacy of the way to derive the formula. We have obtained a similar conclusion, but our underlying physical consideration is somewhat different from the preceding ones.

We should remind ourselves that a 1/2-spin particle at rest performs *Schroedinger's Zitterbewegung*, the amplitude of trembling being a half Compton wave length. The phase factor exp $i(\boldsymbol{K}, \overset{(1)}{\boldsymbol{X}}-\overset{(2)}{\boldsymbol{X}})/\hbar$ appearing in course of calculation describes the effect of the retardation of nuclear field generated by the meson with momentum $\boldsymbol{K}$ which is emitted by the nucleon (2) at the point $\overset{(2)}{\boldsymbol{X}}$ and absorbed by (1) at the point $\overset{(1)}{\boldsymbol{X}}$, and it is this phase factor which yields the spatial dependence of the nuclear potential. Let $\overset{(1)}{\boldsymbol{X}}, \overset{(2)}{\boldsymbol{X}}$ denote the coordinate shifts due to the Zitterbewegung of nucleon (1), (2) respectively. Then for smaller separation of two nucleons, we should take $\tilde{\boldsymbol{X}}$ into account and the phase factor will be given by

$$e^{\frac{i}{\hbar}(K,\, \overset{(1)}{X}-\overset{(2)}{X}+\overset{(1)}{\tilde{X}}-\overset{(2)}{\tilde{X}})}=e^{\frac{i}{\hbar}(K,\, \overset{(1)}{X}-\overset{(2)}{X})}\cdot e^{\frac{i}{\hbar}(K,\, \overset{(1)}{\tilde{X}})}\cdot e^{-\frac{i}{\hbar}(K,\, \overset{(2)}{\tilde{X}})}$$

in first approximation. By virtue of the last two factors the singularity might reduced.

We have calculated the nuclear potential term in the Hamiltonian of two nucleons plus pseudoscalar meson field along the line stated above. (We have employed the pseudoscalar coupling exclusively.) The calculation runs in an analogous way to that in the Pauli-Fierz's solution of the electromagnetic interaction.[2] But we have used here the "physical representation" of the Dirac particles' operators; and consequently the mathematical formulation of the physical idea was much facilitated. The result for the second order of coupling constant is approximately as follows. (The term accompanied with a nucleon pair creation was discarded. M: nucleon mass. μ: meson mass. For simplicity the neutral meson was assumed.) If $r>\frac{\hbar}{MC}$,

$$f^2\frac{e^{-\mu r}}{r}\left\{+(\overset{(1)}{\sigma}\,\overset{(2)}{\sigma})\left(\frac{1}{\mu^2r^2}+\frac{1}{\mu r}-3\frac{(\overset{(1)}{\sigma}r)\,(\overset{(2)}{\sigma}r)}{r^2}\left(\frac{1}{\mu^2r^2}+\frac{1}{\mu r}+\frac{1}{3}\right)\right\}\times\frac{1}{4}\left(\frac{\mu}{M}\right)^2;$$

and when r becomes smaller than the nucleon Compton wave length, the form of potential gets changes, and we have for very small r,

$$f^2\left\{+(\overset{(1)}{\sigma}\,\overset{(2)}{\sigma})\frac{1}{4r}-\frac{(\overset{(1)}{\sigma}r)\,(\overset{(2)}{\sigma}r)}{r^2}\frac{1}{4r}\left(1+2e^{-\frac{\mu}{M}}\right)\right\}.$$

At the beginning of calculation, we have adopted two approximations, the nonrelativistic

approximation and the neglection of all recoil effects: that is, more concretely, (i) the nucleon energy operator is taken as $\left(Mc^2+\frac{P^2}{2M}\right)\rho_3$: (ii) the change of kinetic energy after the virtual meson emission is rejected; (iii) the Zitterbewegung is assumed to be equal to that in case of $p=0$, i.e., $\tilde{X}=\frac{\hbar}{2Mc}\rho_2\sigma$; (iv) the effect due to the coordinate shift (which compensates the decrement of the transversal spin by increasing the orbital angular momentum in order to conserve the total momentum) is neglected throughout.

Fuller account of these points will appear shortly.

The authors wish to express their cordial thanks to Mr. G. Takeda for his kind help in this work.

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Fuller account of the physical representation, including the cases of spin 1 and 0 and photon, will be published shortly in this journal.

On the μ-Meson Decay.

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January 25, 1950

Recent investigations have shown that the μ-meson decay is accompanied by the emission of one electron (or positron) and two neutral particles.[1] Considering the theoretical correlation to β-decay, one of these two neutral particles is supposed to be neutrino, but the identification of another particle still remains unknown.

As to this point we may propose a problem, i.e.; if these two neutral particles are completely identical (that is, if the 2nd neutral particle is also neutrino), there exists a possibility, besides the ordinary spontaneous decay, that the μ-meson decays to only one electron (or positron). Of course, such a process is prohibited for the free μ-meson by the conservation law of energy and momentum. It is allowed, however, for the meson bound in the orbit or subject to scattering in the Coulomb-field of nucleus. In these cases the decay is followed by the emission of an energetic electron inheriting the most part of the μ-meson's energy, If, therefore, it is determined by observation whether these processes exist in nature, we shall have a new knowledge of this unknown neutral particle. For this purpose, we have calculated the probability of aforesaid transition processes and discussed their results. I) As the interaction energy of the μ-meson decay we take the following expression

$$H=g(\psi_e^*\beta\psi_\nu)(\psi_\nu^*\beta\psi_\mu)+\text{c.c.} \qquad (1)$$

where ψ_e, ψ_ν and ψ_μ are the wave function of electron, neutrino and μ-meson respectively. It is for the brevity of the calculation that we have chosen the expression (1). It should be noted, however, that as the interaction energy of the scalar type other than (1), we may take, for instance,

$$H=g(\psi_e^*\beta\psi_\mu)(\psi_\nu^*\beta\psi_\nu)+\text{c.c.}$$

but it prohibits the decay of μ-meson accompanied by only one electron. For in the calculation of matrix element the spur of the factor $(\psi_\nu^*\beta\psi_\nu)$ can be calculated independently of another factor for the case of closed process (i.e. creation and annihilation of the virtual neutrino), and becomes zero. Now we obtain as the probability of the ordinary decay of the μ-meson due to the interaction (1).

$$w_0=(g^2/48\pi^3)\mu(\mu/2)^4 \quad \hbar=c=1,$$
$$\mu:\ \mu\text{-meson mass}$$

II) Next, we may calculate the probability of one electron decay for the case of μ-meson bound in k-orbit of the atom. The result is

$$w=w_0(2^{14}/3)\pi(1/1+(1/\alpha z)^2)^4(\alpha z)^{-3}A, \qquad (2)$$

where α is the fine structure constant, z the

atomic number of the atom and A is given by

$$A=\left|\frac{1}{2\pi^2\mu^3}\int k^2 dk\right|^2 \qquad (3)$$

This divergent integral is due to the momentum of the virtual neutrino in the intermediate state and is of 3rd order divergence. Finally, calculating the probability that the μ-meson decays into one electron (or positron), under the influence of the Coulomb field of the nucleus, we obtain

$$w'=w_0\times 3\cdot 2^{12}\pi^2e^4z^2(1/\beta)(1/\mu^3)A(1/V) \qquad (4)$$

in the approximation $E_0 \gg \mu$. $\beta=P_0/E_0$ is the velocity of the μ-meson and V is the volume in which the wave functions are normalized. When there are N atoms in the unit volume, we have $w''=w'\cdot N$ as the probability for the μ-meson decaying due to the above process in passing through the unit length, while we have

$$w_0'=w_0\sqrt{1-\beta^2}\ /\beta$$

as the probability for the case of the ordinary decay. Therefore, (4) turns out to be

$$w''=3\cdot 2^{12}\pi^2e^4Nz^2(1/\mu^3)A\cdot w_0'/\sqrt{1-\beta^2} \qquad (5)$$

III) The probabilities (2) and (5), in its absolute values, strongly depend on the divergent integral. This divergence may be saved by the regulator subjected to the appropriate conditions. Another device is the introduction of the neutral fields, besides the neutrino field. These remedies, however, are not interesting, if not correlated to the problems of the self-energy and others due to the interaction such as (1). Therefore, we may not go beyond giving the values of the probability on various cut-off limits of the integral A. As seen from the Table (1), these processes are of small effect compared with the ordinary decay process, so far as the integral is cut oft near the μ-meson mass.

	cut-off energy	0.0073_μ	1_μ	2_μ	8.33_μ	$(=1800m)$
w	$(Z=13.7)$	10^{-17}	$4.5\ 10^{-15}$	$2.9\ 10^{-13}$	15	
	$(Z=27.4)$	10^{-12}	$1.5\ 10^{-13}$	0.95	$1.6\ 10^2$	$\times w_0$
w''	$(Z=82)$	—	$2.8\ 10^{-15}$	$1.8\ 10^{-13}$	$9.4\ 10^{-10}$	$\times w_0'\left(\frac{E_0}{\mu}\right)$

We wish to express our hearty thanks to Professor S. Sakata for his suggestions and to Mr. H. Umezawa for many helpfull advices concerning this problem.

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Anomalous Magnetic Moment of the Nucleon.

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The anomalous magnetic moments of the nucleon by vector and pseudovector mesons are calculated according to Feynman-Dyson's formalism. The divergences of mass type, and of charge type can be removed by renormalization. In the vector theory, there appear difficulties in relation to vacuum polarization, but, in the case of the vector coupling, non-gauge invariant terms of them vanish by defining the zero-point energy of meson-nucleon system in the static magnetic field as

$$\frac{-2}{(2\pi)^3}\int d\vec{p}[(\vec{p}+e\vec{A}+f\vec{\phi})^2+\varkappa^2]^{\frac{1}{2}}.$$

In the vector and pseudovector theory, terms containing the tensor coupling give diverging magnetic moments. They converge, however, if we use Pauli's regulator, by the condition $\sum c_i\varkappa_i^2=0$ and $\sum c_i\varkappa_i=0$ in the case of tensor coupling only and cross terms respectively. But, then it also changes the converging integrals. So, we list here the results in the diverging form.

Setting,

$\mu_p=f^2c_f/\pi$, $\mu_n=0$, for neutral theory ;

$\mu_p=2f^2d_f/\pi$, $\mu_n=-2f^2(c_f-d_f)/\pi$, for charged theory ;

we give c_f and d_f for $\delta<2$ as follows. Here, f^2c_f means f^2c_f, g^2c_g, fgc_{fg}, in the case of vector (pseudovector) coupling, tensor coupling, and cross terms respectively, and

it is similar for d_f.

I) Vector theory,

$$c_f=\left(\frac{1}{2}-\delta^2+\delta^2(2-\delta^2)\log\frac{1}{\delta}-\frac{\delta(2-4\delta^2+\delta^4)}{(4-\delta^2)^{1/2}}\cos^{-1}\frac{\delta}{2}\right),$$

$$d_f=c_f,$$

$$c_g=-\frac{1}{\delta^2}\left(\frac{1}{4}-\frac{1}{6}\delta^2(1-5\delta^2+3\delta^4)+\delta^4(1-3\delta^2+\delta^4)\log\frac{1}{\delta}-\frac{\delta^5(5-3\delta^2+\delta^4)}{(4-\delta^2)^{1/2}}\cos^{-1}\frac{\delta}{2}+\int_0^1 x(2-x^2)F(x)dx\right)$$

$$d_g=-\frac{1}{\delta^2}\left(\frac{1}{12}-\frac{1}{2}\delta^2(5-10\delta^2+3\delta^4)-\delta^2(2-15\delta^2+14\delta^4-3\delta^6)\log\frac{1}{\delta}-\frac{\delta^3(16-7\delta^2-4\delta^4+3\delta^6)}{(4-\delta^2)^{1/2}}\cos^{-1}\frac{\delta}{2}-\frac{3}{2}\int_0^1(1-x)(3-7x+9x^2)F(x)dx\right),$$

$$c_{fg}=-\frac{1}{\delta}\left(\frac{8}{3}-\delta^4+\delta^2(5-6\delta^2+\delta^4)\log\frac{1}{\delta}-\frac{\delta(6-13\delta^2+10\delta^4-2\delta^6)}{(4-\delta^2)^{1/2}}\cos^{-1}\frac{\delta}{2}-\int_0^1 x(14-13x)F(x)dx\right),$$

$$d_{fg}=\frac{1}{\delta}\left(\frac{1}{6}-\frac{1}{2}\delta^2(3+\delta^2)+\delta^2(8-7\delta^2+\delta^4)\log\frac{1}{\delta}-\frac{\delta(6-20\delta^2+9\delta^4-\delta^6)}{(4-\delta^2)^{1/2}}\cos^{-1}\frac{\delta}{2}+26\int_0^1(1-x)(2-3x)F(x)dx\right),$$

II) Pseudovector theory,

$$c_f=-\frac{1}{\delta^2}\left(1-\frac{1}{2}\delta^2(13-2\delta^2)-\delta^2(6-8\delta^2+\delta^4)\log\frac{1}{\delta}-\frac{\delta^3(20-10\delta^2+\delta^4)}{(4-\delta^2)^{1/2}}\cos^{-1}\frac{\delta}{2}\right),$$

$$d_f=\frac{1}{\delta^2}\left(1+\delta^2(2-\delta^2)-\frac{1}{2}\delta^2(2-9\delta^2+2\delta^4)\log\frac{1}{\delta}-\frac{\delta(2+12\delta^2-13\delta^4+2\delta^6)}{2(4-\delta^2)^{1/2}}\cos^{-1}\frac{\delta}{2}\right),$$

$$c_g=-\frac{1}{\delta^2}\left(\frac{9}{4}+\frac{1}{6}\delta^2(11+23\delta^2-3\delta^4)-\delta^2(8-21\delta^2+9\delta^4-\delta^6)\log\frac{1}{\delta}-\frac{\delta^3(36-27\delta^2+3\delta^4+\delta^6)}{(4-\delta^2)^{1/2}}\cos^{-1}\frac{\delta}{2}+\int_0^1 x(1-2x)(4-x)F(x)dx\right),$$

$$d_g=\frac{1}{\delta^2}\left(\frac{11}{12}-\frac{3}{2}\delta^2(1+\delta^2)^2+\delta^2(16-37\delta^2+20\delta^4-3\delta^6)\log\frac{1}{\delta}-\frac{\delta(8-62\delta^2+41\delta^4-2\delta^6+3\delta^8}{(4-\delta^2)^{1/2}}\cos^{-1}\frac{\delta}{2}+\frac{3}{2}\int_0^1(1-x)(3-x-9x^2)F(x)dx\right),$$

$$c_{fg}=0,$$

$$d_{fg}=\frac{1}{\delta}\left(\frac{5}{2}-\delta^2-(2-4\delta^2+\delta^4)\log\frac{1}{\delta}-\frac{\delta(8-6\delta^2+\delta^4)}{(4-\delta^2)^{1/2}}\cos^{-1}\frac{\delta}{2}\right),$$

Here δ is the ratio of meson mass to nucleon mass, and

$$F(x)=\log\frac{2w}{M^2\cdot\phi(x)}-\frac{5}{2},\quad \phi(x)=(1-x)\delta^2+x^2,$$

where, M is the nucleon mass. We have used here the "w-method" in integration, and w tends to infinity.

If we consider above results together with Case's results, we can readily see that it is only in the scalar theory that the meson distributes around the nucleon and forms meson cloud, but the effect of the meson cloud has inverse sign with that of expected intuitively. We now consider the results for the vector (pseudovector) coupling only in the vector (pseudovector) theory. Then, in conclusion, calculated magnetic moments do not fit to experimental values, even approximately, in any type of meson. Therefore, if we want to get approximate values in the second order of perturbation theory, it is necessary to mix two or more meson fields. As for mixture fields, furthermore, at least one of scalar or vector meson and another

of pseudoscalar or pseudovector mesons must be used.

I should like to express my gratitude to Prof. S. Sakata and Mr. H. Umezawa for their kind interest in my work.

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The East-West Effect of the Cosmic-Ray in the Upper Atmosphere.

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The symmetry of the east-west intensity of the cosmic-ray in the upper atmosphere[1] ($\gtrsim$1.5mH_2O) gives the serious difficulty against the proton primary hypothesis expected from the experimental results of the east-west effect at the various altitudes from the sea level up to the stratosphere[2] ($\sim$2.5m H_2O). Recently it has been found that the proton is the principal component as yet. In order to make clear the problem of the east-west effect in the upper atmosphere, we calculate the angular distribution of the soft component produced from the presently accepted process,—that is, proton—π-meson—μ-meson—electron.

I) To estimate, phenomenologically, the angular spread between the primary proton and the soft component produced from it adequate to explain the symmetry of the east-west intensity of the cosmic ray in the upper atmosphere, we proceed on the following assumption. (a) The probability that, by the collision with the air nuclei, proton produces π-mesons which give the soft origin through π–μ and μ–e decay processes, depends upon the intensity and the magnetic cut off energy of the proton. The proton intensity decreases in air as $exp\,(-x/l\cos\theta_0)$ (θ_0 is the angle between the direction of the proton and vertical.) (b) Thus produced soft-component traverses in air cascade-typically. The multiplication function will be determined phenomenologically to agree with the experiments of the soft component in the upper atmosphere. (c) The angular distribution between the primary proton and the produced soft origin is assumed as

$$d\Omega\, R(\theta)=\begin{cases}\text{const} & (|\theta|\leq\omega)\\ 0 & (|\theta|>\omega).\end{cases} \qquad (1)$$

Thus, comparing our calculation based on (a), (b) and (c) with the experimental value of the east-west effect, $\lesssim$10%, which is given by Johnson-Barry, we obtain as the value of ω (at the atmopsheric depth =0.5m H_2O)

$$\omega\sim0.87 \quad \text{radian.} \qquad (2)$$

This estimation is carried on the soft-component only, while the experiment by Johnson is on the total intensity. As the hard-component is considered to retain its direction more strongly than the soft-component, so the value obtained in (2) may be the minimum one.

II) Next, we estimate theoretically the angular distribution of the soft origins which are produced at the atmospheric depth xm H_2O by the proton which falls vertically from the top of the atmosphere. For this case, i) the emission of π-meson is assumed to be angularly uniform in the mass-system of the nucleon-nucleon collision. ii) The angular spreads associated with the processes of π–μ and μ–e decay are taken into consideration.

Now, we calculate the energy intensity of the soft origin, $E_s(x,\theta)$, which is produced at the atmospheric depth xm H_2O, and then we define the mean value of the angular spread of the soft-component as

$$\langle\theta_s\rangle=\int|\theta|\,E_s(\theta,x)d\theta\,/\int E_s(\theta,x)d\theta. \qquad (3)$$

Calculating the mean value of ω in the meaning such as (3), we obtain

$$<\omega>_{exp}=\int^{w}|\theta|d\theta \;/\; \int^{w}d\theta. \qquad (4)$$

The same is done with the μ-meson intensity,

$$<\theta_\mu>=\int|\theta|\mu(\theta, x)d\theta \;/\; \int\mu(\theta, x)d\theta.$$

These results at the atmospheric depth $=0.5\text{m}\ H_2O$ are listed in Table (I).

$<\theta_s>$	$<\theta_\mu>$	$<\omega>_{exp}$	
0.197	0.155	0.435	radian

Table I. (geomagnetic latitude $\lambda=20°$N)

As seen from the Table I, the angular spread which is estimated by our calculation, gives only a half of that experimentally required. And from the comparison between $<\theta_s>$ and $<\theta_\mu>$, it becomes clear that the influence of the decay process to the angular spread is small.

The absolute intensity of the soft-component in the upper atmosphere, as is well known, cannot be explained by the soft-component produced only from the μ-meson decay. However, if we assume that the soft-components which are produced by other causes (e.g. neutral meson decay) are emitted with the angularly uniform distribution in the mass-system of the nucleon-nucleon collision, the symmetry of the east-west effect in the upper atmosphere cannot be also explained. The effective component which has the possibility to explain this symmetry, seems to be the heavy nuclei in the primary. However, this component is also insufficient, as far as we assume the multiple production of π-meson by nucleon-nucleon collision. It is rather expected that π-mesons are produced in the plural processes, or in the anisotropic distribution in the mass-system of nucleon-nucleon collision.

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A Picture for Cosmic-Ray Stars.

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Excellent photographic experiments carried out by Bristol Group[1] give us ample knowledges about the nuclear interaction, since they visualize the nuclear interaction more directly and cover the wider range of energies than others. In order to facilitate our understanding on this problem, we must, first of all, build a picture or a model for cosmic-ray stars from this experiments. In other words, we ask the kind and the energy of their agents.

We may tentatively classify the nucleons producing the stars into A-, B-. and C-component, according to their genetic relations and energies.[2] We see in what follows this classification is convenient and consistent for various cosmic-ray phenomena.

A-componont. This component has the energy of 2 BeV or greater and responds to the stars with $n_s \geq 2$.[3] The abundance ratio of neutrons and protons (we call this quantity as n–p ratio) in this component is about 1/2 in the stratosphere. These excessive protons clearly correspond to the surviving primary protons. Thus the energy of this component must be higher than the knee of the primary spectrum$\sim$2 BeV[4]. The integral n_s-distribution of the stars is roughly expressed by the power-law n_s^{-2} for $n_s=2\sim10$, and seems to reflect the energy spectrum E^{-2} of this component. Considerable part of this tracks in these stars will be mesons produced by A-nucleons.

B-component. This component has the energy between 1 and 2 BeV, and responds

mostly to the stars with $n_s=1$. The n–p ratio of this component is always about 1 in the stratosphere and also at the mountain altitude, which means that the B-component is produced from A-component. The direction of the motion seems to be nearly conserved in this A–B transmutation. 0_p, 1_n, and 1_p stars in the stratosphere have nearly equal absolute frequency and similar size distribution. This suggests that they have the same nature and the charge exchange probability in a nuclear collision is nearly 1/2. The precise comparison of the frequency of 0_p and 1_p stars and the distribution of the scattering angle in 1_p stars tell us that about 20% of thin secondary tracks in 1_p stars are meson. Applying the single scattering formula to this angular distribution and considering the energy transferred to the nucleus, we can estimate the energy of the B-nucleons. This result does not contradict our presumption. Furthermore, the energy loss in one collision can be shown nearly 1/3.

C-component. This component has the energy below 1 BeV, and responds to the stars with $n_s=0$. It is produced by A- or B-component. The angular distribution of grey tracks in a star tells us that the angular spread of A–C or B–C transmutation is considerably wide and may be approximated by the isotropic distribution. C-protons have the non-relativistic velocity, and thus undergo large ionization loss. In fact the observed n–p ratio of the stars with $n_s=0$ both in stratosphere and at mountain altitude is larger than 1. Most of the 0_n stars can be interpreted as all energy of the incident neutron is dissipated into secondary grey and black protons and neutrons. Thus the energy spectrum of the C-neutrons can be obtained from the size-distribution of 0_n stars, and it turns out to be nearly E^{-2}.

We shall be able to discuss the various properties of the nuclear interaction on the above picture as will be accounted in detail in the later issue.

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3) We use the same notations as used by Bristol group, see ref. 1). n_s is the number of thin secondary tracks in a star, e.g. 2_s means the star with $n_s=2$. The suffix in 2_p or 2_n represents that the agent is charged or neutral.
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Diffusion of the Nucleon Component.

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Our classification of the nucleon component in A-, B-, and C-component[1] allows us to treat their diffusion through the atmosphere. The comparison of this calculation with the experimental data will testify our simple picture.

In the course of calculation, we make the following assumptions which are approximately verified by various cosmic-ray observations.

1) The absorption coefficient of each component is $\sim 1/125\,\text{g cm}^{-2}$ air.

2) The collision mean free path for nuclear interaction is nearly the geometrical one i.e. $\sim 100\,\text{g cm}^{-2}$ in emulsion.

3) The probability for charge exchange in one nuclear collision is $\sim 1/2$.

4) The direction of the motion is conserved in A-B transmutations, whereas A–C and B–C transmutations take place approximately isotropic.

The results of our analysis are as follows.

Stratosphere ($45\,\text{g cm}^{-2}$). From the ex-

perimental data of Bristol group[2], we get the following table:

n_s	0	1	≥ 2
n-p ratio	5.45 ± 0.87	0.91 ± 0.23	0.57 ± 0.19
absolute freq ency (per c.c., day)	794	186	108

The n–p ratio of A component is calculated from 1), 2), 3) as 0.50. If we take the primary cosmic-ray intensity to be 0.12 $\sec^{-1}$ cm^{-2} $sterad^{-1}$,[3] the total intensity of A component at 45 g cm^{-2} is $A=2.67\times10^4$ day^{-1} cm^{-2}. Thus the frequency of A-stars should be 1.06×10^3/c.c. day. On the other hand, we get from the above table the maximum estimation of A-stars as 3.3×10^2/ c.c. day, i.e. 2_s star$+2\times(60\%$ of 1_s star), considering the n–p ratio of 1_s stars. The factor 2 comes from our assumption 3). This large discrepancy will be plausibly due to an overestimation of the primary proton intensity, or missing of the stars with two or less heavy tracks or both. In fact, nearly half of the single counting rate at the free atmosphere was reported due to soft component[4], though there may be some question. The large amount of back rays[5], i.e. slow electrons, protons and mesons, and the considerable percentage of α-particles in primary rays[6] will make lower the primary proton intensity than that previously supposed.

The frequency of B-stars are estimated to be 1.5×10^2/c.c. day, i.e. $2\times(40\%$ of 1_s stars). Thus the total intensity of the B-component is B=0.45A. This means the multiplicity in A–B transmutation to be ~0.32.

Subtracting A- and B-stars we have the frequency of C-stars as 6.1×10^2/c.c. day.

Mountain altitude (690 g cm^{-2}). The experiments of Bristol group reports[7],

n_s	0	1	≥ 2
n-p ratio	7.10 ± 0.43	1.14 ± 0.15	0.76 ± 0.13
absolute frequency (per c.c., day)	9.36	0.98	0.49

From the experimental data in stratosphere, we get the total intensity of A-component as $A=8.7$ day^{-2} cm^{-2} at this altitude, and the frequency of A-stars should be 0.35/c.c. day. Thus stars with $n_s\geq2$ can be regarded as A-stars.

The frequency of B-stars is estimated as 1.9/c.c. day, i.e. $2\times(1_s$ stars). From this, the total intensity of B-component turns out to be B=5.4A. Thus we get the multiplicity in A–B transmutations as 0.49, and this figure does not contradict with the above estimation.

Subtracting A- and B-stars, we have the frequency of C-stars as 8.4/c.c. day..

C-stars. The analysis of the C-stars contains some ambiguity, as the C-component is the secondary or tertiary product in our picture, and the ionization loss of C-protons can not be neglected. Therefore, we will here take only the C-neutrons into account. If we neglect B–C transmutations and consider only A–C transmutations, the multiplicity in A–C transmutation becomes 1.7 or 1.4 from the analysis of stratosphere data or mountain altitude data, respectively. This multiplicity may be an overestimation, since B–C transmutations will make considerable contribution at the mountain altitude, and the C-secondaries from primary α-particles must not be negligible in the stratosphere.

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Nucleon-Nucleon Interaction at High Energies.

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February 7, 1950

Using appropriate unique potential, we can explain all the problems concerning the neutron proton system with energy below 90 MeV[1]. Further knowledges on nuclear forces in higher energy region are obtained from the photographic observation of cosmic-ray stars[2]. We find from them the following characteristic features of the collisions of high energy nucleons with atomic nuclei[3].

1) The cross-section for the nuclear interaction is nearly equal to the geometrical cross-section of the collided nucleus.

2) The momentum distribution of the knocked-out protons, which are observed as grey tracks, is roughly represented by $P^{-3}dP$.

3) For incident nucleon with energies below 2 MeV, nucleon-nucleon collisions are almost elastic, i.e. the collisions associated with meson production will be $\sim 20\%$.

4) The fractional energy loss in one nuclear collision is $\sim 1/3$ and nearly independent of the incident energy.

5) The probability for the charge exchange in one nuclear collision is $\sim 1/2$.

These results allow us infer the nucleon-nucleon interaction at high energies.

Previously, we calculated the problem on the penetration of fast nucleons in the nuclear matter[4]. In this case, we adopted as the nuclear force the central Yukawa potential, which was well fitted to the low energy (≤ 90 MeV) neutron-proton scattering and the deuteron binding energy. The main result of this calculation is that the cross-section is nearly proportional to $(c/v)^2$, thus to $1/E$ in non-relativistic energy region, and tends to $4.7 \times 10^{-27}\,\mathrm{cm}^2$ per one nucleon in a nucleus at high energies.

This contradicts with 1). At first, we supposed that the large total cross-section at high energies, i.e. 1), might be mainly due to the inelastic scattering. But now, 3) tells us that the probability for meson production is not so great that the opacity of the nuclear matter for high energy nucleons must be due to the increase of the elastic scattering cross section itself.

Although our calculation is based on the central Yukawa Potential, this conclusion will not be changed, even if we introduce some tensor forces. This can easily be seen from the Fourier component of the tensor Yukawa field in higher momentum region, which has the same property as that of the central one. The singular Yukawa potential $exp(-\varkappa r)/r^3$ may be rejected by 2) and also by Berkeley scattering experiments.

Thus we are compelled to suppose some new type of the nuclear forces (e.g., velocity dependent forces as expected in the case of pseudoscalar meson field) sets in at high energies, and makes the total cross-section larger.

The charge dependency of the nuclear forces can not be concluded from 3), since a nucleon will make plural collisions in a nucleus and any charge dependency may be masked.

The nuclear forces at high energies may be largely dependent on the type of mesons assumed. Full account will soon be published in this journal.

1) R. S. Christian and E. W. Hart ; unpublished. The authors are indebted to Dr. L. Baumhoff, who sent to one of us (Y. Y.) a copy of their work.

2) The experiment of Bristol group. See our preceding notes.

3) See our preceding notes.

4) Y. Fujimoto and Y. Yamaguchi : Prog. Theor. Phys. in press.

Note on the Interaction of Meson with Nucleon.

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February 13, 1950

Following the preceding notes[1] we can infer the considerably weak intersection of π-meson with nucleon by the reasons mentioned below. Several arguments for this presumption are described in a separate paper.[2] Here we confine ourselves in the problem of the meson production in cosmic rays, especially referring to a photographic work of Bristol group.[3]

In the first place we take our attention to the fact that the frequency distribution of n_s reflects the energy spectrum of incident nucleons up to $n_s=10$. Such a relation would not take place, if the observed thin tracks were mainly the mesons produced multiply in one nucleon-nucleon collision and the multiplicity of the produced mesons slowly increased with the incident energy, It seems to be plausible that the thin tracks are mesons and protons produced by the plural act in the concerning energy region, $\lesssim 20$ BeV. In fact, Heitler and Janossy succeeded to explain this behaviour by their plural production theory.[4]

In spite of the above fact, the mesons are likely produced multiply in higher energy region. A direct evidence is obtained from the Rochester star, which is caused by the incident energy about 10^{12} eV per nucleon[5]. It is a very interesting question in what height of energy the genuine multiple production sets in. Unfortunately, there seems to be so few materials to solve this problem, but we may mention the cloud chamber experiment of Fretter.[6] The energy of the incident nucleons giving rise to the penetrating showers is approximately estimated as 5 BeV or greater from the absolute frequency.* The integral frequency distribution of shower size is represented as nearly n^{-3} for $n \gtrsim 4$, where n is the number of shower particles. This distribution corresponds to neither the energy spectrum of agent nucleons nor the plural production by Heitler and Janossy. In the latter theory much steeper decrease is expected for larger n. This may mean that the genuine multiple production seems to set in for the showers with several or more mesons in heavier nuclei. Thus the multiple process for the interaction of meson with nucleon will be appreciable for greater energy than ever expected.

The energy where the multiple process gets over the single one will give a clue to find the magnitude of the interaction of meson with nucleon. The order of this energy may roughly be estimated by comparing the magnitudes of the matrix elements of the first and the second orders and the density factors of final states. The ratio of the cross sections for both processes is about $g(E/\mu)\rho$, provided that the interaction contains the spatial differentiation of meson field, where g, μ and ρ mean the dimensionless coupling censtant, the mass of meson and the ratio of density factors, respectively. If the multiple process overcomes at $E/\mu \sim 10$, we get $g^2 \sim 10^{-3}$ to 10^{-2}. which is smaller the facter ten or more than the current value.

The magnitude of g^2 can also be estimated from the ratio of the elastic to the meson producing scatterings for the energy of nucleon about $g^2(E/\mu)\rho'$ times of that for the elastic scattering. This leads to the same order of the magnitude of g^2, considering that about 80% of the nuclear collisions in th s energy region are elastic.

These considerations will in detail be discussed lately.

* It is supposed that the mixed showers caused by neutral mesons are frequently observed in such a condition that the energy of the agent nucleons is considerably high. The other experiments accompanied by fewer mixed showers may correspond to lower

energy events.
1) Y. Fujimoto *et al*: in this issue.
2) H. Fukuda, S. Hayakawa and Y. Miyamoto: Prog. Theor. Phys. in press.
3) See the references of the preceding notes.
4) W. Heitler and L. Janossy: Proc. Phys. Soc.
5) M. F. Kaplon, B. Peters and H. L. Bradt: Phys. Rev. **76** (1949), 1835.
6) W. B. Fretter: Phys. Rev. **76** (1949), 511.

Relativistic Covariance in the Quantum Electrodynamics

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February, 8, 1950

Various researches[1), 2)] have recently clarified that both electromagnetic and C-mesonic self energies of a moving electron have the transformation properties of inertia mass. Relativistic covariance of the present quantum electrodynamics is guaranteed only on the basis of the point electron model. But in the calculation for the self energy of an electron, on account of its divergence, we must cut off the momentum l of virtual particles at any large value and then pass to the limit of $l\rightarrow\infty$. This procedure is contradictory with the point model of an electron. Such an inevitable procedure, which contradicts with the postulate of the quantum electrodynamics, gives the finite self stress of an electron and destructs the covariance of the field theory. This situation has been made clear by the recent discussion of Pais and Epstein.[3)]

Accordingly, we can say that, if a theory satisfying the following two conditions, can be found within the present field theory, it must be just a correct theory of the point electron model:

(I) The self energy of an electron has the transformation property of inertia mass.

(II) The self energy of an electron is finite.

Now, we have the C-meson theory as satisfying conditions (I) and (II). We can therefore expect that, applying the C-meson theory, the self stress of an electron becomes zero and its stable particle aspect can be given.

Our aim is just to verify this expectation. We have calculated the self stress by C-meson field of an electron at rest with an extended Pais' method.

In our case, the energy-momentum tensor is given by

$$T_{\mu\nu}=T_{\mu\nu}^{e}+T_{\mu\nu}^{E.M.}+T_{\mu\nu}^{c},$$

$$T_{\mu\nu}^{e}=-\frac{1}{4i}\left(\bar{\psi}\partial_{(\mu}\gamma_{\nu)}\psi-\partial_{(\mu}\tilde{\gamma}_{\nu)}\bar{\psi}\psi\right),$$

$$T_{\mu\nu}^{E.M.}=-\frac{1}{4\pi}\left(F_{\mu\rho}F_{\nu\rho}-\frac{1}{4}\delta_{\mu\nu}F_{\rho\sigma}^{2}\right)$$

$$T_{\mu\nu}^{c}=-\frac{1}{4\pi}\left\{\frac{\partial\phi}{\partial x_\mu}\frac{\partial\phi}{\partial x_\nu}\right.$$

$$\left.-\frac{1}{2}\delta_{\mu\nu}\left(\frac{\partial\phi}{\partial x_\lambda}\frac{\partial\phi}{\partial x_\lambda}+\varkappa^2\phi\phi\right)\right\}, \qquad (1)$$

where $\partial_{(\mu}\gamma_{\nu)}=\partial_\mu\gamma_\nu+\partial_\nu\gamma_\mu$, $\partial_\mu=\frac{\partial}{\partial x_\mu}-ieA_\mu$, $F_{\mu\rho}=\frac{\partial A_\rho}{\partial x_\mu}-\frac{\partial A_\mu}{\partial x_\rho}$, $T_{\mu\nu}^{e}$, $T_{\mu\nu}^{E.M.}$ and $T_{\mu\nu}^{c}$ are the energy-momentum tensors for an electron field ψ, an electromagnetic field A_μ and a C-meson field ϕ respectively and $\varkappa$ is a C-meson mass.

Taking the trace of (1), we get

$$\int T_{\mu\mu}dv=\mu\int\psi^*\beta\psi dv-f\int\psi^*\beta\psi\phi dv$$

$$+\frac{1}{4\pi}\int\frac{\partial\phi}{\partial x_\lambda}\frac{\partial\phi}{\partial x_\lambda}dv+\frac{\varkappa^2}{2\pi}\int\phi\phi dv,. \quad (2)$$

From a simple consideration, we get for the expectation value of $T_{\mu\mu}$ in the second approximation of the perturbation theory

$$<\int T_{\mu\mu}dv>=<\mu\int\psi^*\beta\psi dv>$$

$$+<\frac{\varkappa^2}{4\pi}\int\phi\phi dv>. \qquad (3)$$

Denoting the self stress, mass and the self energy for an electron at rest by $S(0)$, μ and $\delta\mu$ respectively, we have

$$<\int T_{\mu\mu}dv>=3S(0)+\mu+\delta\mu.$$

Then considering (3), we get

$$S(0)=-\frac{1}{3}\left\{\delta\mu-\mu\frac{\partial(\delta\mu)}{\partial\mu}\right.$$

$$\left.-<\frac{\varkappa^2}{4\pi}\int\phi\phi dv>_{p=0}\right\}, \qquad (4)$$

where $\delta\mu=\delta\mu^{E.M.}+\delta\mu^{c}$, $\delta\mu^{E.M.}$ and $\delta\mu^{c}$ are respectively the electromagnetic and the C-mesonic self energy of an electron at rest, and therefore $\delta\mu$ is finite by the condition $f^2=2e^2$. Moreover we find that $<\frac{\varkappa^2}{4\pi}\int\phi\phi \times dv>_{p=0}$ is equal to $\delta\mu-\mu\frac{\partial(\delta\mu)}{\partial\mu}$ in the second approximation and $S(0)$ becomes zero.

In this way, we can verify that the correct point model of an electron in the quantum electrodynamics is guaranteed by the C-meson theory and an electron has a stable particle aspect. Moreover, we find that our expectation that, a theory, satisfying two conditions (I) and (II), must be a correct theory of the point electron model, is true.

We would like to thank Professor S. Sakata for his kind interest and Mr. R. Kawabe for his invaluable discussions.

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Derivation of the Interaction Potential from Field Theory

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February 16, 1950

From the standpoint of the quantum field theory, the ordinary potential of interacting particles regarded as function of their space coordinates and other variables is neither accurate nor sufficient. The relativistic effect, or the retardation on one hand, and the quantum effect, or the recoil and higher order force on the other, must be taken into account to give a complete description of the interaction. This is especially important in the meson theory where the field has mass and charge and the coupling is strong. Such a treatment is possible for scattering and other perturbation problems concerning elementary systems by using the S-matrix expansion. In this case the potential seems to be a rather clumsy tool to make use of. Up to now, however, the latter combined with the Schrödinger eigenvalue equation is an indispensable and inevitable one for dealing with bound systems, though a relativistic many body problem is rightfully hoped to be established for analyzing the difficulties regarding the deuteron and other nuclei. Here we shall be content only with re-examining the old potential concept and improving it so as to conform more closely to the rigorous field theoretical view. The usual second order potential is easily derived from the Tomonaga-Schwinger theory as has been done by many authors.[1] It is given by

$$H_2(1,2)=-\frac{1}{2}\varepsilon^2 j_\lambda(1)\int\bar{\Delta}(12)j_\lambda(2)dt_2, \qquad (1)$$
$$H_2=H_2(1,2)+H_2(2,1),$$

where λ denote all indices other than the space coordinates. In the non-relativistic ($c=\infty$) approximation, this gives a unique potential. But when we consider the retardation correction, there arise ambiguities, as was recently discussed by Araki[2] and Toyoda.[3] For example, we cannot decide between (1) and

$$H_2'=-\varepsilon^2\int j_\lambda\left(1,t+\frac{t'}{2}\right)\bar{\Delta}(12)$$
$$j_\lambda\left(2,t-\frac{t'}{2}\right)dt', \qquad (2)$$

which do not give the same result for interaction potential. Moreover, when applied to the electromagnetic field, neither of them reproduces the well established Breit interaction. (Here we are not using the elimination of the longitudinal field.) But the latter

is provided by the formula (putting $\varepsilon j_\lambda=$ charge density)

$$H_2''=j_\lambda(1)V_2^{(0)}(12)j_\lambda(2)-\frac{dj_\lambda(1)}{dt}$$

$$\times V_2^{(2)}(12)\frac{dj_\lambda(2)}{dt},$$

$$V_2^{(0)}=-\frac{\varepsilon^2}{4\pi}\frac{e^{-\mu r}}{r},$$

$$V_2^{(2)}=\frac{1}{2\mu}\frac{\partial}{\partial\mu}V_2^{(0)}, \qquad (3)$$

$$\left(\text{or } \frac{r^2}{2}V_2^{(0)} \text{ for } \mu=0\right)$$

which agrees with the former only in those matrix elements where energy is conserved. It seems that there is no reason to prefer one to others, so long as we are restricted to the second order potential. But the problem is at once settled when we require that *the potential should be so determined that it gives correct results for the S-matrix up to the fourth (and possibly, higher) order terms.* Indeed, looking apart from self-energy like corrections, there are two main diagrams (a) and (b) describing the fourth order interaction. One is a mere iteration of second order interaction while the other is an essentially new

process which cannot be expressed in terms of H_2. The difference between the true fourth order terms and the terms obtained by iterating the second order retarded potential must then be regarded as the fourth order potential H_4 which, in cooperation with H_2, yields the correct result for S_4. An approximate calculation, using v/c expansion and omitting delta-function like terms (effective only for distances of the order of the particle Compton wave length) gives the following result:

$$H_4=-\left(\frac{\varepsilon^2}{4\pi}\right)^2\cdot\frac{1}{\pi r}K_0(2\mu r)[j_\lambda(1),j_\rho(1)]$$

$$\times[j_\lambda(2),j_\rho(2)]-\frac{1}{4}V_2^{(0)}V_2^{(2)}$$

$$\left[\frac{d}{dt}(j_\lambda(1)j_\rho(2)),j_\rho(1)j_\lambda(2)\right],$$

K_0=modified Bessel function of order 0 (4)

when j_λ is a static $(\sim(v/c)^0)$ quantity, and

$$H_4=\left(\frac{\varepsilon^2}{4\pi}\right)^2\frac{1}{4\pi}\frac{\mu}{m^2r^2}K_1(2\mu r)$$

$$[j_\lambda(1),j_\rho(1)][j_\lambda(2),j_\rho(2)] \qquad (5)$$

when j_λ is non-static $(\sim v/c)$. Thus, when there are non-commuting spin indices in the interacting quantity, the fourth order potential is not zero even in the limit of instantaneous propagation of the field, as will easily be understood by inspecting the diagram (a) and (b). When j is static and commutable, as is the case of the fourth component of the Møller interaction, we can eliminate the second term of (4) by a canonical transformation independent of past history:

$$\Psi=S\Psi_1,$$

$$S=\exp\left[-\frac{i}{4}\frac{d}{dt}(j(1)j(2))V_2^{(2)}\right]$$

At the same time the second order potential (2) receives an additional term which just brings it to the desired form (3). This example will warn us of too naive an attitude toward the familiar potential concept. The relation between the potential which should be used for solving the Schrödinger equation and the interactions appearing in the successive terms of the S-matrix is not so simple as one may think, even if the high singularities and divergences which appear in the extreme relativistic region are disregarded. In

the meson theory, in particular, where occurs the non-commuting isotopic spin variable, there may be an essential difference in the higher order potentials depending on whether the theory is neutral, charged, or symmetric. This circumstance will be interpreted physically as the effect of stringent correlation between the quanta emitted and absorbed successively as a result of the charge conservation, a situation encountered also in the multiple production of mesons by nucleons.

More detailed account will be given in a later report.

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Van Hove, Phys. Rev. **75** (1949), 1519
Watson and Lepore, Phys. Rev. **76** (1949), 1157.
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3) Toyoda, Soryûsiron Kenkyû **1** (1949), No. 3, 129. (in Japanese).

On the Second Maximum of the Rossi Transition Curve: I.

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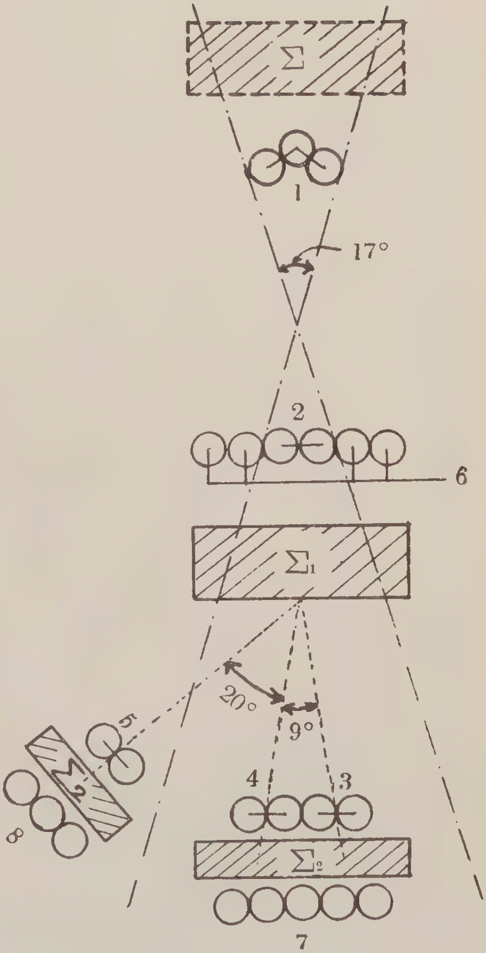
February 16, 1950

Many papers on the second maximum of the Rossi transition curve have hitherto been published, some of them confirming, while others denying its existence.

We have been performing some measurements on the same problem. In order to avoid the ambiguity in the interpretation of our results, the following precautions were taken in the experiments: (a) only those showers are counted, which are produced in a layer of lead by charged particles incident from vertical direction with narrow angle. (b) Air showers and local showers produced by surrounding material are removed.

The arrangement is shown in Fig. 1. The counts of showers containing two or more particles produced in lead Σ_1, (1, 2, 3, 4,

Fig. 1.



-6) and (1, 2, 4, 5, -6), are measured when the trays 1, 2, 3, 4 and 1, 2, 4, 5 discharge simultaneously, but anti-tray 6 does not discharge. The average angle of showers of the former is 9° and that of the latter is 20°. These showers are denoted by NS and WS, respectively. Similarly, the showers containing three or more rays, (1, 2, 3, 4, 5, -6), are denoted by NWS.

The transition curves obtained for NS, WS and NWS are shown in Fig. 2. They lead us to the following conclusion: The conspicuous maximum is found, their position

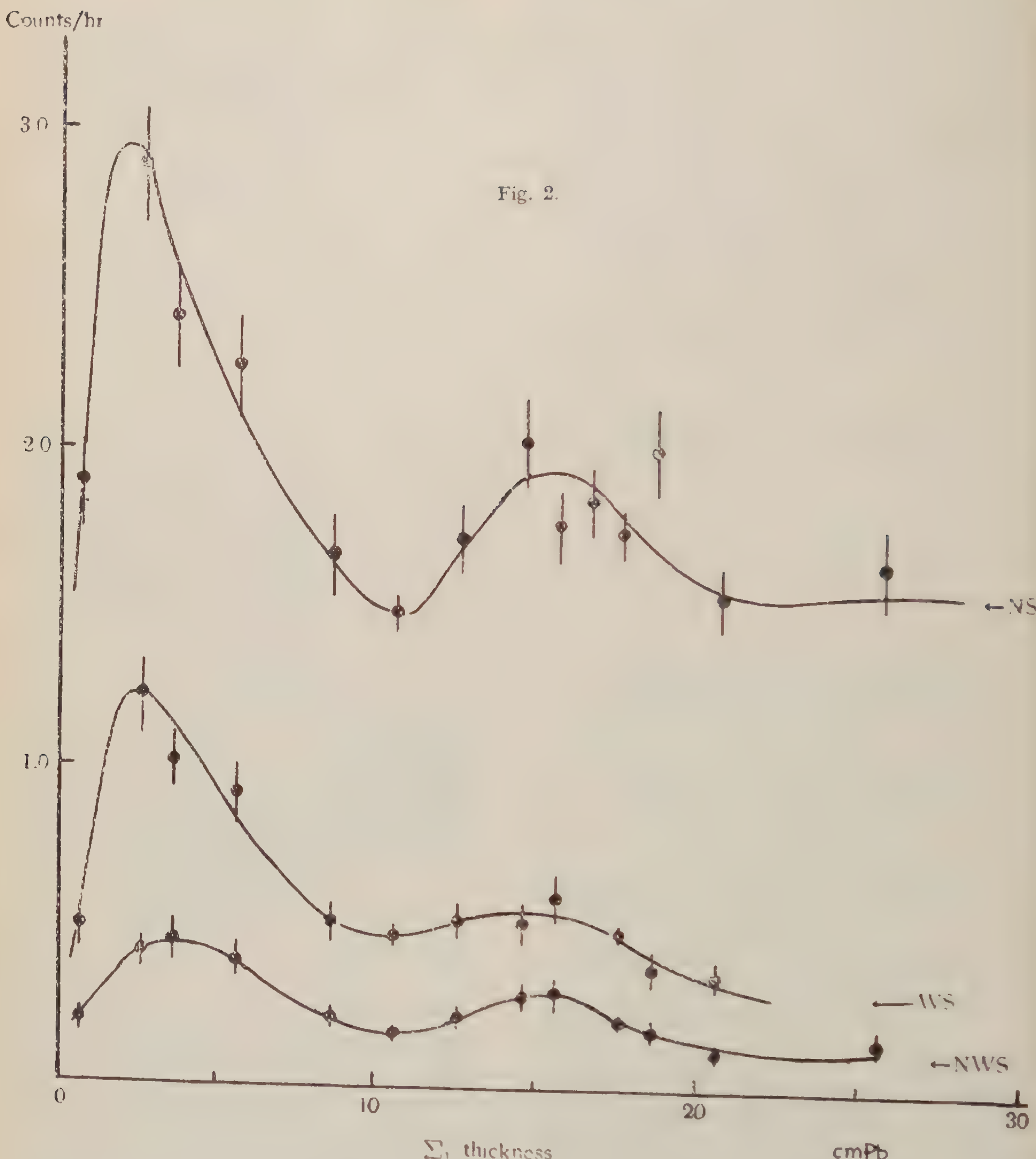
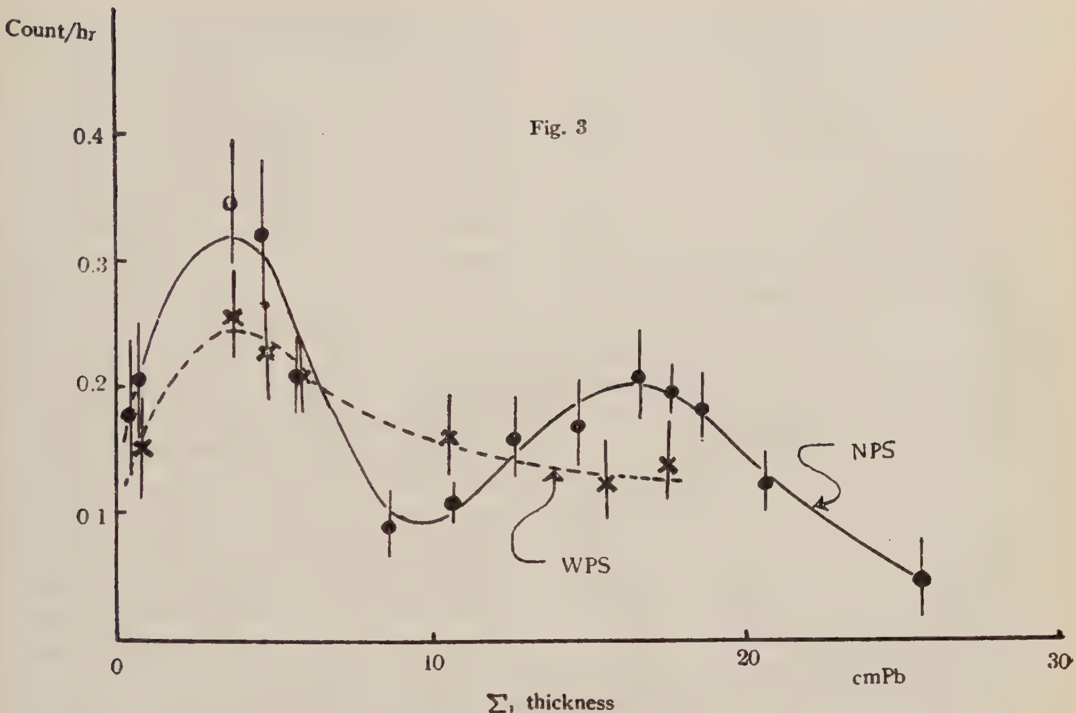


Fig. 2.

being between 15∼16 cm Pb for NS. The slight maxima exist near 15∼16 cm Pb for WS and NWS.

To study the penetration of shower rays, 10 cm lead absorber $\sum_2$ was interposed between shower detecting trays 3, 4, 5 and the lowest trays 7, 8, and we measured the countrng rate of (1, 2, 3, 3, 7, −6) which was the counts of NS accompanied by the discharge of two or more counters in tray 7, denoted by NPS, and that of (1, 2, 4, 5, 8, −6) which was WS accompanied by the discharge of one or more countes in tray 8, denoted by WPS. The results obtained are



shown in Fig. 3. From the figure two maxima are recognized near 5 cm Pb and 15 cm Pb of $\sum_1$ for NSP, while the maximum of WSP is found near 5 cm Pb of $\sum_1$ but that near 15 cm Pb is not clear.

On the Second Maximum of the Rossi Transition Curve: II.

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February 16, 1950

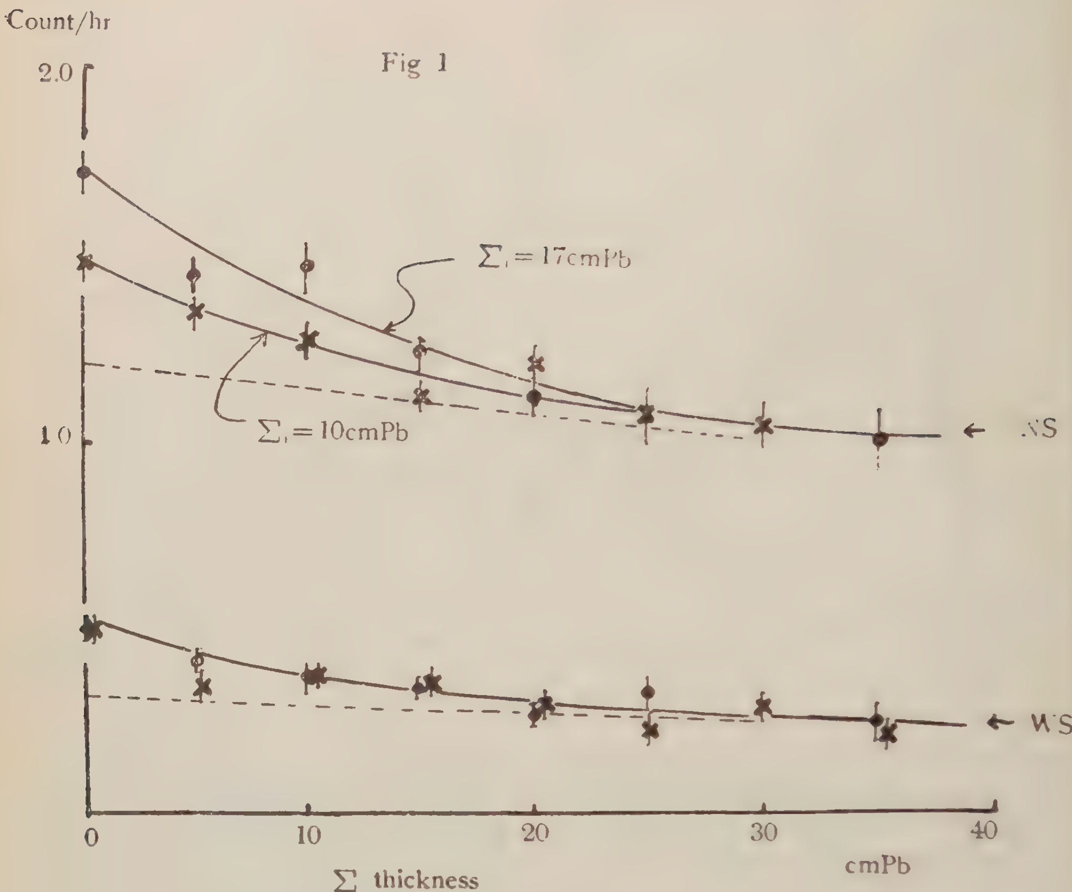
In order to study the origin of the second maximum of the Rossi transition curve, described in Part I, we measured the absorption of shower producing rays by varying the thickness of absorber $\sum$ (Fig. 1 of I), when we interposed 10 cm and 17 cm lead as $\sum_1$.

The results are shown in Fig. 1. The dotted lines in the figure show the estimated background of knock-on electrons. (1)

Subtructing the background and assuming the exponential absorption of the remaining showers, we obtained the following values as the mean free path in lead of shower producing rays $\lambda_{Pb} \approx 150$ g/cm^2, in both cases of $\sum_1 = 10$ and 17 cm Pb in NS. Similarly, $\lambda_{Pb} \approx 150$ g/cm^2, in the case of $\sum_1 = 10$ or 17 cm Pb in WS.

These values are in good agreement with that of the mean free path of the rays producing penetrating showers. (2) Therefore, we conclude that the rays which cause the second maximum of the Rossi curve are identical with those which produce the penetrating showers, probably nucleonic component.

The frequency of showers at the second maximum, corrected for the background, is about 0.1 % of the particles incident on $\sum_1$. This rate agrees fairly with number of fast protons at sea level. (3)



Next we studied the dependence of the occurrence of showers containing penetrating particles on shower producing material Σ_1, by counting (1, 1, 3, 4, 5, 8, -6) in the same weight of paraffin and lead. From it we conclude that the occurrence of such showers is consistent with what is expected from the geometrical cross section of nucleus of material in Σ_1.

Full reports will be published lately.

The authors wish to thank Dr. Nishina and Mr. Miyazaki for their interest and encouragement and Mr. H. Takeuchi for his help throughout the progress of the work.

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1) G. Cocconi: Phys. Rev. **75** (1949), 1074; **76** (1949), 984.

3) B. Rossi: Rev. Mod. Phys. **20** (1948), 537.

Interpretation of the Second Maximum of Rossi Curve.

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February 16, 1950

Since long ago, it has been obscure whether the second maximum of Rossi curve really exists or not. Recent experiment carried out by Kameda and Miura seems to establish

this evidence[1]. Clay has also get the similar result, but attempted to explain it by the knock-on electrons of mesons[2]. On the contrary, the former authors infer the second maximum as caused by the nucleon component, on the basis of the absorption law of primary rays and of the initial increase in lead and paraffin. But they do not enter the deeper consideration on the mechanism of this phenomenon.

As far as we concern to the experiment of Kameda and Miura, Clay's experiment may not be accepted, because, in the first place, the primary rays posess the characteristic nature of nucleons as seen in II, and in the second place, the saturation should take place much faster than at the position of the second maximum by the following reason : The intensity of hard rays penetrating through their counter train amounts only a few hundreds times of the frequency of showers at the second maximum. Then the average energy of these mesons is, in contrast to the case of Clay, so low that the saturation takes place in the very small thickness of $\sum_1$. Furthermore, the second maximum for wide angle showers can not be attributed to knock-on showers. The contribution from knock-on secondaries, though estimated in II, may not be free from ambiguity, so that the absorption coefficient of the primary rays can not be literally accepted. The stronger arguments will be required to establish the nature of the second maximum.

We notice the similarity between the second maximum of curve A and the first maximum of curve D in I. The knock-on electrons may less contribute to the penetrating coincidences represented as D and E because of the small probability of double knock-ons. The curve A minus the knock-on part should, therefore, be similar to curve D. Curve D has another maximum at the same thickness of $\sum_1$. We interpret such behaviour as follows referring to the lemma in the subsequent paper[3].

A primary nucleon produces a shower containing two kinds of shower particles. One has the limited range as large as 17 cm Pb and the other undergoes the exponential absorption. The former secondaries produce a maximum at the total thickness of $\sum_1$ and $\sum_2$ 17 cm Pb. Another maximum due to the latter ones is not displaced by the presence of $\sum_2$. Thus the second maximum of curve A is composed of these two kinds of secondaries and $\sum_2$ plays a role to discriminate between them. The somewhat steeper decrease after the first maximum of D than that of the second maximum of A seems to support this interpretation, since in A the superposition of both secondaries may smear out the steepness. However, there remains an objection that the decrease after these maxima is steeper than the decrease of primary rays, though they should be equal. This may be due to the effect of the geometrical condition such that the counters 3,4 and 7 do not cover the sufficient area to detect the all secondary particles.

The presumption that the penetrating shower has two kinds of secondary rays has already been pointed out by Walker[4]. The qualitative interpretation such as they are composed of protons and pi-mesons is in agreement with ours, but the range of the shorter secondaries is much longer than that obtained by him. In the concerning experiment, however, the energy of incident nucleons is estimated as about 2 Bev or greater from the absolute intensity. Then the shower contains two or more fast particles capable to penetrate the 10 cm lead layer[5]. Among these particles about a half may be protons which mainly undergo ionization loss. Our interpretation is, therefore, supposed to be not far from reality. These considerations will be testified by the experiment varying the thickness of $\sum_2$.

Lastly we consider the reason why the second maximum distinctly appears in the experiment under consideration. In this ap-

paratus the solid angle of the counter train is very small. This brings about the considerable decrease of the intensity of mesons and electrons which obey cosine square angular distribution, whereas the nucleons are less reduced because of their steep angular dependence, presumably $\cos^8$ law. If this were not the case, the tail of the cascade shower and the background of knock-on electrons would mask the second maximum. Thus the small solid angle is essential for this experiment, associated with the reason mentioned by Kameda and Miura. Further the selection of penetrating showers must not be too strict, since the strict selection will considerably reduce the intensity of available nucleons so that the second maximum will vanish.

Full account will be published in Jour. Sci. Res. Inst. The present authors thank to Messrs. Kameda and Miura who kindly gave us their unpublished data and contributed to our work by valuable advice.

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3) J. Nishimura: in this issue.
4) W. D. Walker: Phys. Rev. in press.
5) Y. Fujimoto and S. Hayakawa: Prog. Theor. Phys. in press.

A Remark on the Transition Curve of Cosmic Ray Showers

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February 16, 1950

There is a characteristic feature in the transition curve of penetrating showers, which appears in the Rossi curve measured by Kameda and Miura[1]. The showers are generated in a layer with the thickness x_1 and their secondary rays penetrate through another layer with the fixed thickness x_0. The transition curve taken by varying x_1 shows two distinct maxima as seen in curve D of Fig. 3 in I. To interpret this behaviour we consider a feature of the transition curve as a lemma.

The frequency of showers in our case is represented by

$$N(x_1)=\int_0^{x_1} dx e^{-\lambda x} g(x_0+x_1-x) \times E(R-x_0-x_1+x), \qquad (1)$$

where $e^{-\lambda x}$ represents the intensity of primary rays at the thickness x and the function gE means the intensity of secondary rays with maximum range R. The function $E(x)$ has the values 1, 1/2 or 0 according to x positive, zero or negative. Transforming the integral variable (1) is reduced to

$$N(x_1)=e^{-\lambda(x_0+x_1)}\{F(x_0+x_1)-F(x_0)\}, \qquad (1')$$

where

$$F(x)=\int^x dy e^{\lambda y} g(y) E(R-y). \qquad (2)$$

The position of the maximum of $N(x_1)$ is obtained by the relation

$$\lambda-\lambda\frac{F(x_0+x_1)}{F(x_0)}+\frac{F'(x_0+x_1)}{F(x_0)}=0. \qquad (3)$$

Here we must define the differentiation of $F(x)$ as

$$F'(x)=e^{\lambda x}g(x)\{E(R-x)+\delta(R-x)\}, \quad (4)$$

where δ is the δ-function.

Case 1. $R=\infty$. The curly bracket in (4) can be dropped and $F(x)$ is put $H(x)$. If the position of the maximum is independent on x_0 which is the case in the second maximum of curve D, $H(x)$ must be an exponential function as seen from (3). Then $g(x)$ is constant of exponential, but the former is ruled out because it can not produce any maximum.

Case 2. R is finite. Then it is inconvenient to use $F'(x)$ because of the presence of improper δ-function. We see the behaviour of the curve directly from $N(x_1)$ making use

of the expression

$$F(x)=H(x)E(R-x)+H(R)E(x-R). \tag{5}$$

We get the expressions for $N(x_1)$ corresponding to three different ranges R:

$$N(x_1)=0 \qquad \text{for} \quad R\leq x_0, \tag{6a}$$

$$N(x_1)=e^{-\lambda(x_0+x_1)}\{H(R)-H(x_0)\}$$
$$\text{for} \quad x_0\leqq R\leqq x_0+x_1, \tag{6b}$$

$$N(x_1)=e^{-\lambda(x_0+x_1)}\{(x_0+x_1)-H(x_0)\}$$
$$\text{for} \quad x_0+x_1\leqq R. \tag{6c}$$

The maximum can only exist for $R\geqq x_0+x_1$. In (6b) $N(x_1)$ is monotonic decreasing, so that the maximum occurs at the smallest value of x_1, namely, $x_1=R-x_0$. In (6c) the solution depends on the functional form of $H(x)$. A special case is described in Case 1.

As a simple example, which will be the current case, we put

$$g(x)=e^{-\mu x}. \tag{7}$$

Then the maximum exists at

$$x_{max}=ln(\lambda/\mu)/(\lambda-\mu), \tag{8}$$

where μ must be as small as to be $x_{max}\leqq R-x_0$. In the special case, $x_0=0$, (6c) turns into a monotonic increasing function and there should be

$$x_{max}=R-x_0. \tag{8'}$$

One must notice $x_{max}\leqq R-x_0$ in any case.

When the absorber $\sum_2$ is removed (see Fig. 1 of I), $x_0=0$, the position of the maximum is shifted to larger x by x_0 in Case 2, whereas no change takes place in Case 1, because $R-x_0$ is always infinity irrespective to the different finite values of x_0. This explains the behaviour of curve A in Fig. 2 of I.

1) T. Kameda and I. Miura; in this issue, Part I being cited as I.

The Conduction Electron with Low Energy in Ionic Crystals*)

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February 17, 1950

In ionic crystals, the interaction between the conduction electron and ions is strong. On account of it, the mean free path of the conduction electron in ionic crystals is smaller than that in metals[1]. And there is a question about the validity of the time-dependent perturbation method used for the calculation of the mean free path. The mean free path (or the mobility) of the conduction electron plays an important role in various phenomena concerning electron conduction and etc. Especially, in the phenomena of the dielectric breakdown, for which the theory has not yet been established, it is very important[2]. The interaction between the conduction electron and ions in ionic crystals gives a theoretical possibility of the so-called Landau's self-trapped electron[3]. But it has not yet been observed experimentally. These circumstances show that the reinvestigations for the behavior of the electron with low energy in ionic crystals is necessary.

Investigation of the interaction energy shows that the Fröhlich and Mott's formula for the interaction energy W and the frequency of optical longitudinal modes ν must be modified as follows,

$$W=1/\varkappa_0 W', \quad \nu^2/\nu_l^2=\varkappa/\varkappa_0, \tag{1}$$

where ν_l is the Reststrahl frequency, $\varkappa$ and $\varkappa_0$ are the static and optical values of the dielectric constant, and W' is the Fröhlich and Mott's formula for the interaction energy. The self-trapped state is obtained from the Hamiltonian of the system consisting of an electron and optical longitudinal modes of the lattice vibration under the adiabatic potential approximation. Its thermal activation energy E_0 and optical activation energy ε_0 are

$$E_0=13.53\left(\frac{5}{16}\right)^2 Z_0^2\, e.V., \quad \varepsilon_0=3E_0\, e.V.,$$
$$Z_0=1/\varkappa_0-1/\varkappa$$

If we consider the atomic structure of lattice, these values are modified as follows

$$E_0=1.802\, Z_0^2\, e.V., \quad \varepsilon_0=5.406\, Z_0^2\, e.V.$$

That the self-trapping potential behaves Z_0e/r at a large distance from the trapping center is closely connected with the interaction energy (1).

Investigation of the interaction energy indicates that the free electron approximation is not suitable to the conduction electron with low energy and that in such a case the wave function of the electron corresponding to each momentary configuration of the lattice vibration must be used. If we denote the wave function of the self-trapped state containing the lattice vibration, which trapping center is at the lattice point $\boldsymbol{r}_a$, by $\varphi(\boldsymbol{r}-\boldsymbol{r}_a, x)\Theta^{r_a}(x)$, where x is the variable describing the lattice vibration, the wave function satisfying the lattice periodicity as above mentioned may be constructed as follows,

$$\Psi_K(\boldsymbol{r})=C\sum_{ra}\exp(i\boldsymbol{K}\boldsymbol{r}_a)\varphi(\boldsymbol{r}-\boldsymbol{r}_a, x)\boldsymbol{\Theta}^{r_a}(x),$$

where C is the normalization factor. The energy $\boldsymbol{E}(\boldsymbol{r})$ for the $\Psi_K(\boldsymbol{r})$ is calculated with some approximations. Its result gives for NaCl, when $|\boldsymbol{k}|$ is small, $m/m^*\simeq 5\cdot 10^{-2}$ where m is the electron mass and m* is the effective electron mass. The mobility of the conduction electron with low energy may be calculated with this Ψ_K. Its calculation is in progress.

A detailed account of this work will be publisched in the Science Reports of the Tôhoku University.

* This paper has many similar points with the lecture by Mr. R. Kubo at the annual meeting of the Phys. Soc. of Japan on April 28, 1949, and the writer is indebted to Mr. R. Kubo for indications.

1) H. Fröhlich and N. F. Mott, Proc. Roy. Soc. (A) **171** (1939), 496.
F. Seitz: Phys. Rev. **76** (1949), 1376.
2) For instance, see A. von Hippel and R. S. Alger: Phys. Rev. **76** (1949), 127.
3) L. Landau: Phys. Zeits. d. Sowjet. **3** (1933), 664.
R. W. Gurney and N F. Mott: Proc. Phys. Soc. **49**, extra part, (1937), 32.
J. J. Markham and F. Seitz: Phys. Rev. **74** (1948), 1014.

On the Conception of the Energy Band in the Perturbed Periodic Potential

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There are many physical significances in the electronic states, where a slowly varying potential field is superposed to the periodic field of the crystalline lattice. But many of foregoing theories are obliged to rely upon the semi-classical or model-like intuitions, on account of the complexities of the mathematical treatments. We have tried some mathematical treatments of this problem using the method of Bloch's tight binding approximation. We have investigated the behaviors of the wave functions and the state density in such a field, and have given the clear interpretation to " the deformed energy band," which has been sometimes used without any interpretation.

Now we assume the following approximate expression for the wave function of the whole crystal,

$$\psi=\sum_l C_l\phi_l,$$

where ϕ_l and l denote the wave function of the isolated atom and the number of the lattice point respectively. We can see the behavior of the wave function from the de-

pendency of C_l to l, which modulates the atomic wave functions. From the fact that the superposed potential slowly varying, we obtain the following difference equation for C_l,

$$C_{l+1}+2(\epsilon-v_l)C_l+C_{l-1}=0,$$

where ϵ and v_l are the representation of the energy and the superposed potential field respectively, with the appropriate conversion of the unit.

We have solved the above equation in the general forms for the two cases where $v_l \propto l$ and $v_l \propto l^2$, and determined the values of ϵ (the allowable values of the energy) from the quantum mechanical conditions. When $v_l \propto l$, C_l becomes the cylindrical function, $C_l = J_{l-l_0\epsilon}(l_0)$, where l_0 is a suitably chosen parameter with the dimension of the length. When $v_l \propto l^2$, our treatment becomes rather complicated. Fourier transformation, W.K.B. approximation and saddle-point method are employed in our culculations.

First we investigate the behavior of C_l as the function of l with the fixed value of ϵ. It is the common feature for both two cases that C_l shows the oscillating form only in a certain interval of l, and damps off exponentially both outside of it for suitably selected values of ϵ. Next we set up the rectangular coordinate in the plane, whose ordinate and abscissa represent ϵ and l respectively, and bound the region, where C_l (namely ψ) with the eigenvalue corresponding to the each point of the ordinate shows the oscillating form, and define this region "the energy band" of the crystal under the superposition of any potential field. It is also mathematically proved that the energy band of the periodic field is shifted up- or downward for each value of l by the amount v_l in the presence of the superposed field. Our one-dimensional treatments are easily extended to the three dimensional model and suggest that the above mentioned feature is general for any form of v_l. These results make clear the conception of "the deformed energy band," which has been conventionally introduced without any definite interpretation.

A detailed account of this work is published in the Science Reports of the Tôhoku University.

Recently, the authors knew the papers about the similar treatments by Slater[1] and James[2], but could not read their papers till the authors' paper was prepared.

1) J. C. Slater, Phys. Rev. **76** (1949), 1592.
2) H. M. James, Phys. Rev. **76** (1949), 1602 1611.

Note on the Elimination of the Normal-dependent Part from the Hamiltonian

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February 22, 1950

In general, Hamiltonian density is decomposed into two parts:

$$H(x)=H_{\text{I}}(x)+H_{\text{II}}(x)\,, \qquad (1)$$

$$H_{\text{I}}(x)=-L(x),$$

$$H_{\text{II}}(x)=K_{\mu\nu}(x)N_\mu N_\nu\,. \qquad (2)$$

We make S-matrix using such a Hamiltonian, but the normal dependent part seems to play no role, so let us consider a method of the elimination of $H_{\text{II}}(x)$ by modifying Dyson's P-bracket[1].

Now we construct a family of space-like surfaces, and write $\sigma_1 > \sigma_2$ to express that σ_1 lies in the future of σ_2.

The P-bracket in the integrand of S is defined by

$$P(H(x_1)H(x_2)\cdots H(x_n))$$
$$=H(x_{i_1})H(x_{i_2})\cdots H(x_{i_n}) \qquad (3)$$

for 0) $\sigma(x_{i_1}) > \sigma(x_{i_2}) > \cdots > \sigma(x_{i_n})$,

where $x_{i_1}, x_{i_2}, \cdots$ are the permutation of $x_1, x_2, \cdots$.

But how shall we define it, if some of the

surfaces $\sigma(x_1)$, $\sigma(x_2)$,$\cdots$ coincide with each other? For instance,

i) $\sigma(x_{i_1})=\sigma(x_{i_2})>\sigma(x_{i_3})>\cdots>\sigma(x_{i_n})$,

ii) $\sigma(x_{i_1})=\sigma(x_{i_2})=\sigma(x_{i_3})>\cdots>\sigma(x_{i_n})$,

....

In the ordinary theory, it was not necessary to define P for these cases. Because $P(H(x_1), H(x_2),\cdots H(x_n))$ is the integrand of $4n$-dimensional integral while the domain for the case i) is $(4n-1)$-dimensional, and $(4n-2)$-dimensional for ii), and so on, i.e., the measure of the domains of i), ii),$\cdots$ is zero, so that these cases do not contribute to the integral. We will admit such a discussion for the cases ii), iii),$\cdots$, but will consider more deeply about the case i), then P is decomposed as follows:

$$P(H(x_1)\cdots H(x_n))=P(H(x_{i_1})H(x_{i_2}))\times P(H(x_{i_3})\cdots H(x_{i_n})).$$

Our problem is to define $P(H(x_{i_1}), H(x_{i_2}))$, while generally the following formula holds:

$$P(H(x)H(x'))=(1/2)\{H(x)H(x')\}+(1/2)\varepsilon(x,x')[H(x),H(x')], \tag{4}$$

where

$$\varepsilon(x,x')=\begin{cases}1, & \text{for } \sigma(x)>\sigma(x').\\ -1, & \text{for } \sigma(x')>\sigma(x).\end{cases}$$

The case when $\varepsilon(x, x')$ troubles us is $\sigma(x)=\sigma(x')$, because in this case ε is not defined. If $[H(x), H(x')]=0$, the indefiniteness of $\varepsilon(x, x')$ does not trouble us, but if $[H(x), H(x')]\gtrless 0$, then it will be of the following form;

$$[H(x),H(x')]=\frac{2\hbar}{i}K_{\mu\nu}(x,x')\frac{\partial^2\Delta(x-x')}{\partial x_\mu\partial x_\nu'},$$

where $K_{\mu\nu}(x, x)=K_{\mu\nu}(x)$. Consequently $P(H(x)H(x'))$ is given by

$$P(H(x)H(x'))=\frac{1}{2}\{H(x)H(x')\}+\varepsilon(x,x')\frac{\hbar}{i}K_{\mu\nu}(x,x')\frac{\partial^2\Delta(x-x')}{\partial x_\mu\partial x_\nu'}. \tag{4'}$$

Here, in order to avoid the indefiniteness of of P, we define the modified P-bracket P^*

$$P^*(H(x)H(x'))=\frac{1}{2}\{H(x)H(x')\}+\frac{\hbar}{i}K_{\mu\nu}(x,x')\frac{\partial^2\bar{\Delta}(x-x')}{\partial x_\mu\partial x_\nu'}, \tag{5}$$

then $\bar{\Delta}(x-x')=\varepsilon(x,x')\Delta(x-x')$ does not depend on the family, and P^* coincides with P if $\sigma(x)\gtrless\sigma(x')$.

The general definition of P^* is as follows:

case 0) $P^*(H(x_1)H(x_2)\cdots H(x_n))$
$=P(H(x_1)H(x_2)\cdots H(x_n))$.

case i) $P^*(H(x_1)H(x_2)\cdots H(x_n))$
$=P^*(H(x_{i_1})H(x_{i_2}))P(H(x_{i_3})\cdots H(x_{i_n}))$.

For the cases ii), iii),$\cdots$, P^* is left undefined, because they do not contribute to the integral.

By this modified P^* we can prove the following formula:

$$\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{-i}{\hbar}\right)^n\int_{\sigma_0}^{\sigma}dx_1\int_{\sigma_0}^{\sigma}dx_2\cdots\int_{\sigma_0}^{\sigma}dx_n\times P(H(x_1)H(x_2)H(x_n))$$
$$=\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{i}{\hbar}\right)^n\int_{\sigma_0}^{\sigma}dx_1\int_{\sigma_0}^{\sigma}dx_2\cdots\int_{\sigma_0}^{\sigma}dx_n\times P^*(L(x_1)L(x_2)\cdots L(x_n)).$$

The last expression is just the same with that proposed by Koba[2], and is the generalization of Matthews' treatment[3].

1) F. J. Dyson: Phys. Rev. **75** (1949), 486.
2) Z. Koba: Prog. Theor. Phys. **5** (1950), 139.
3) P. T. Matthews: Phys. Rev. **76** (1949), 678.

The Effect of Exclusion Principle on the Nucleon-nucleon Scattering in Nuclear Matter

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February 25, 1950

Adopting the degenerate Fermi-gas model

for nuclear matter, we must take into account the effect of exclusion principle due to the Fermi-distribution of nucleons in nuclear matter, when we calculate the cross section for high energy nucleon-nucleon scattering occurring in nuclear matter. Such effect was discussed by various authors (Heisenberg[1], Serber[2], Goldberger[3], and Rosenfeld[4]).

In order to interpret this effect, we consider the following simple example, which have been discussed by Goldberger[3]: We assume that the differential cross section $\sigma_t/4\pi$ of nucleon-nucleon collision per unit steradian is independent of both the incident energy and the scattering angle in centre of momentum system. Let $E_1{}^0$, $E_2{}^0$, E_1, E_2 denote the energies (in laboratory system) of two nucleons 1 and 2 before and after collision, respectively, (1: incident nucleon, 2: nucleon in nuclear matter before collision). Only the collisions which satisfy the condition (1) can take place;

$$E_1{}^0,\ E_1,\ E_2 \gtrsim E_F, \quad \text{and } E_2{}^0 \lesssim E_F, \tag{1}$$

where

$$E_F = \text{Fermi energy}.$$

The total collision cross section σ_{Nucl} in nuclear matter can be calculated by the method described by Goldberger. One finds:

$$E_1{}^0 \gtrsim 2E_F: \quad \sigma_{Nucl} = \sigma_t\left[1 - \frac{7}{5}\frac{E_F}{E_1{}^0}\right], \tag{2}$$

$$E_1{}^0 < 2E_F: \quad \sigma_{Nucl} = \sigma_t\left[1 - \frac{7}{5}\frac{E_F}{E_1{}^0} + \frac{2}{5}\left(2 - \frac{E_F}{E_1{}^0}\right)^{5/2}\right] \tag{3}$$

σ_t means the total cross section for free nucleon-nucleon scattering. The cross section $\sigma_{Nucl}(E_1{}^0, \varepsilon)d\varepsilon$ that the incident nucleon with energy $E_1{}^0$ loses its energy by an amount between ε, $\varepsilon + d\varepsilon$ in one collision is given by

$$E_1{}^0 - E_F > \varepsilon > E_F:$$

$$\sigma_{Nucl}(E_1{}^0, \varepsilon)d\varepsilon = \sigma_t \frac{d\varepsilon}{E_1{}^0}, \tag{4}$$

$$E_F > \varepsilon > 0: \quad \sigma_{Nucl}(E_1{}^0, \varepsilon)d\varepsilon$$

$$= \sigma_t\left[1 - \left(1 - \frac{\varepsilon}{E_F}\right)^{3/2}\right]\frac{d\varepsilon}{E_1{}^0}. \tag{5}$$

If the collision takes place in free space, the cross section for energy loss reduces to

$$E_1 > \varepsilon > 0: \quad \sigma_{Free}(E_1{}^0, \varepsilon)d\varepsilon = \sigma_t \frac{d\varepsilon}{E_1{}^0}. \tag{6}$$

Only the equation (2) of the above expressions (2)~(5) has ever been derived by Goldberger[3]. In actual cases the cross section for nucleon-nucleon scattering is not spherically symmetric as assumed above, and the effect of the exclusion principle is much larger than the above model. (For the case of high energy nucleon-nucleon scattering by Yukawa-type interaction, see ref. 5)).

The same effect gives rise to the complicated modification of the cross section for meson-production by particles or gamma-rays with energies not much larger than the meson-threshold. Therefore, we must not overlook this effect, when we compare the theoretical expectations with experimental data. These subjects will be discussed in subsequent letters.

* Now at Tokyo University, Tokyo.

1) W. Heisenberg, Leibziger Ber., **89** (1937), 369.
2) R. Serber, Phys. Rev., **72** (1947), 1114.
3) M. L. Goldberger, Phys. Rev., **74** (1948), 1268.
4) L. Rosenfeld, *Nuclear Forces*, Vol. II, (1949).
5) Y. Fujimoto and Y. Yamaguchi, Prog. Theor. Phys., **5** (1950), 76.

On the Anomalous Magnetic Moment of Nucleon in Vector and Pseudovector Meson Theory

S. Hori and K. Sawada

Department of Physics, Kyoto University

March 24, 1950

Recent development of invariant formalism has made it possible for us to calculate

A Note on the C-Meson Hypothesis

T. Kinoshita

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March 25, 1950

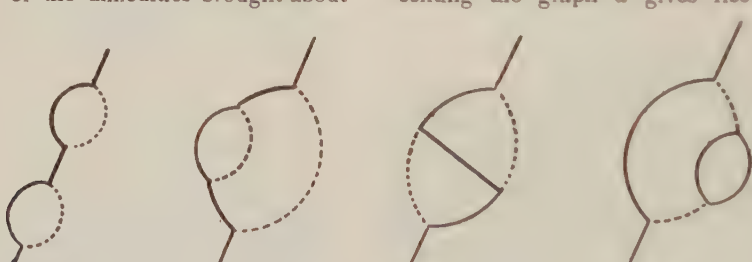
Several years ago, Sakata[1] and Pais[2] proposed independently a device which is effective in making finite the second order self-energy of an electron calculated on the basis of the positron theory. They assume that electrons are necessarily surrounded by a neutral scalar meson field—the C-meson field—besides the ordinary electromagnetic field. It is found that when the coupling constant f between C-meson field and the electron field satisfies the condition

$$f^2=2e^2 \qquad (1)$$

the second order self-energy of an electron becomes finite, and thus the scalar field has some feature of cohesive force acting on the charge of the electron.

In spite of the difficulties brought about by the introduction of a new particle (the C-meson) associated with this field, various attempts have since made to justify this hypothesis both theoretically and experimentally. This method of mixing fields have been extended to analyze the difficulties of the present quantum theory of fields, especially the problem of vacuum polarization[3]. On the other hand, it has also been remarked frequently that the C-meson itself can hardly be observed in the present experiments if it exists at all. Therefore it may not be useless to ask once again whether or not the C-meson hypothesis is a satisfactory one from the theoretical view point. The fourth order self-energy of an electron, for example, may thus give a crucial test for the validity of this hypothesis with which we shall be concerned in this note.

Diagrams corresponding to the fourth order self-energy of an electron due to the interaction with the surrounding electromagnetic field and the C-meson field are drawn immediately as follows:



a b c d

These are second order corrections to the graph of the second order self-energy. Here the dotted line represents either a virtual photon or a virtual C-meson and we must of course take account of all possible diagrams of orders e^4, e^2f^2, and f^4. Matrix elements associated with them are evaluated by the usual method described by Feynman and Dyson[4]. (It is convenient to make use of the relativistic cut-off of Feynman in actual computations.)

Direct calculation of the matrix representing the graph a gives rise to the contribution $\delta\kappa_4{}^a$ to the fourth order self-energy of an electron;

$$\delta\kappa_4{}^a=\frac{1}{0}\;\frac{1}{2\kappa}\,(\delta\kappa_2)^2+B\delta\kappa_2 \qquad (2)$$

where the first term is linearly divergent for *free* electrons and does disappear if the self-mass is amalgamated with the mechanical mass. B is a logarithmically divergent term of the charge renormalization type. $\delta\kappa_2=\delta\kappa_2{}^e+\delta\kappa_2{}^f$ is the self-mass of an electron in the second order which is known to be finite on employing the condition (1). $\delta\kappa_2{}^e$ and

A Note on the C-Meson Hypothesis

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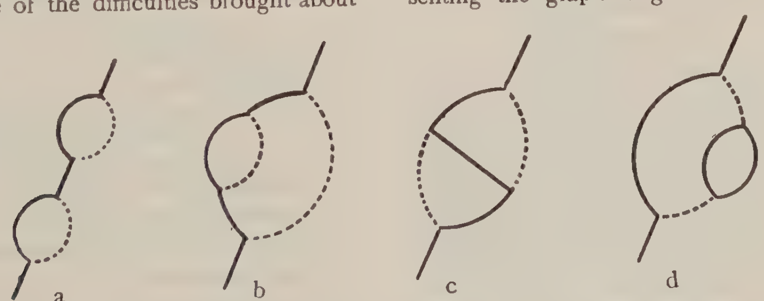

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$$\delta\varkappa_4{}^a=-\frac{1}{0}\frac{1}{2\varkappa}(\delta\varkappa_2)^2+B\delta\varkappa_2 \qquad (2)$$

where the first term is linearly divergent for *free* electrons and does disappear if the self-mass is amalgamated with the mechanical mass. B is a logarithmically divergent term of the charge renormalization type. $\delta\varkappa_2=\delta\varkappa_2{}^e+\delta\varkappa_2{}^f$ is the self-mass of an electron in the second order which is known to be finite on employing the condition (1). $\delta\varkappa_2{}^e$ and

$\delta\kappa_2{}^f$ are electromagnetic and C-mesonic self-masses of an electron, respectively.

The contribution coming from the graph b is of the form

$$\delta\kappa_4{}^b = A\delta\kappa_2 + B\delta\kappa_2 + C\kappa \qquad (3)$$

where A diverges logarithmically and is removed by the mass renormalization. C is finite. There arises thus no difficulty in diagrams a and b if mass and charge are renormalized to the second order. The necessity of mass renormalization in the case of the C-meson theory was once stressed by the present author[5].

The third graph c shows rather different features from the above ones. It has the form

$$\delta\kappa_4{}^c = B'\delta\kappa_2{}^e + B''\delta\kappa_2{}^f + D\kappa \qquad (4)$$

where B', B'', and D are logarithmically divergent. The charge-renormalization factors in (4) does not cancel those of (2) and (3) contrary to the case where the electromagnetic field only is taken into account. This is related to the circumstance that in our case the meaning of the word "correction to a vertex" is not unique at all as in the usual electrodynamics.

After the leading terms are removed by the mass renormalization (of the C-meson field alone) and the charge renormalizations (of both kinds) the modified D_F-function gives rise to a logarithmically divergent term belonging to the graph d. This term arises due to the polarization of the electronic vacuum and therefore it may not be curious even if this is divergent in the C-meson theory.

Collecting the above results, we find that the condition (1) can not insure the convergence of the fourth order self-energy and thus success of the C-meson hypothesis in the second order approximation is no more than an accident. The reason for the existence of the C-meson is thus very much weakened, though one can not exclude the possibility that it still describes a sort of particle which really exists in nature. It must however highly be appreciated that the C-meson theory has been instrumental in analyzing the present complicated features of the theories of elementary particles.

This consideration strongly suggests that various theories of mixing fields, especially those which have been developed relating to the divergences of vacuum polarization, are to be scrutinized by similar methods as sketched here. The regularization procedure of Pauli and Villars may not be rejected[6], since it concerns only with a single type of fields contrary to the "realistic" theories in which mixtures of various types of fields are considered. It is desirable to make these circumstances clearer from the general point of view.

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Note on Five-Dimensional Space and the Self-Energy of Electron.

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April 1, 1950

We pointed out recently[1] that the modified five dimensional theory of Møller's type could accomplish the same role as "regulator" of Pauli and Villars. There we considered it to be an approximate formulation of future

five dimensional theory in correct sense and neglected the five dimensional covariance in its standpoint. If we adhere to the five dimensional covariant formulation, however, it is not correct to take the theory of Møller's type[3]. In this meaning we can take the formalism proposed by Watanabe[4] some years ago. In his theory, however, masses of particles which take parts in processes behave as fifth momentum so that the mass conservation law comes into appearance in the same meaning as the conservation of momentum and energy. Therefore Watanabe's theory cannot be a correct one as it is not profitable in the problems to which particles with their proper masses concerns. But it will be to some extent a good approximate theory if we treat the problems in the frame of electrodynamics excluding the meson field.

As stated in previous paper, we know nothing about the character of the fifth coordinate, nor has been constructed the correct theory which determines *a-priori* masses of particles in nature. Though this fact is the point at which the five dimensional theories should aim, it is just the reason too why these theories could not argue men into compliance.

In the following we shall overlook these defects and calculate the self energy of electron standing on the viewpoint of setting the mass value of initially existing particles *a posteriori*.

Let the equations of motion for electron field and electromagnetic

$$\bigcirc U_\alpha=0, \quad (a=1,\,2,\,3,\,4,\,5) \qquad (1)$$

$$\epsilon_\alpha\frac{\partial\psi}{\partial x_\alpha}=0, \qquad \frac{\partial\tilde{\psi}}{\partial x_\alpha}\epsilon_\alpha=0,$$

respectively, where

$$\epsilon_\alpha=(-i\gamma_5\gamma_\mu,\,\gamma_5), \qquad (2)$$

$$\tilde{\psi}=i\psi^+\gamma_5.$$

As the commutation rules we put

$$[U_\alpha(x),\,U_\beta(x')]=i\delta_{\alpha\beta}\Box(x-x'),$$
$$\{\psi(x),\,\tilde{\psi}(x')\}=\frac{1}{i}\epsilon_\alpha\frac{\partial}{\partial x_\alpha}\Box(x-x'). \qquad (3)$$

The generalized Schrödinger equation is

$$i\frac{\delta}{\delta\Sigma}\Psi[\Sigma]=[-ie\tilde{\psi}\epsilon_\alpha\psi U_\alpha]\Psi[\Sigma], \qquad (4)$$

where, as before, we take the normal of the surface Σ perpendicular to the fifth direction.

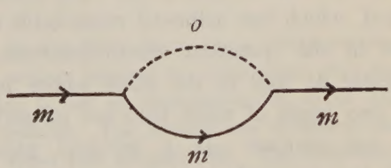
Computing the self energy following Schwinger, we obtain

$$\phi(x)=\text{const.}\int dx'\int dk\,dk'(\exp[i(k+k')(x-x')])[\epsilon_\alpha(ik_5'\epsilon_5)\epsilon_5]\left(\frac{\delta(k^2)}{k'^2}+\frac{\delta(k'^2)}{k^2}\right)\phi(x).$$

If we appoint the mass in this place we get

$$\delta m\psi(x,\,m)=\text{const.}\int(dp)\exp(ipx)\delta(p_5+m)\times(dq)\int_0^1 3i(1-u)\epsilon_5 p_5\delta'(q^2+u^2p^2)\times\psi(p)=0.$$

Hence the self energy of electron vanishes exactly. This situation can be best understood in the Feynman-Dyson's diagram. As is shown in these figures our theory mixes the masses of electron and electromagnetic field variously in the intermediate state, and this circumstance is



usual theory

our theory

just equivalent to take Pauli's subtraction fields into account.

The other problems concerning with these formulation will be investigated in other places.

In conclusion we should like to express our hearty gratitude to Prof. S. Watanabe for his kind interest and advice during this work.

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Radiative Correction to Decay Processes. II.

Beta Disintegration of Nucleon.

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The second order radiative correction to decay processes has been investigated with an attempt to see if the self-consistent subtraction method, which has achieved remarkable successes in the quantum electrodynamics, is applicable as well to the cases where more than two kinds of wave field are interacting with one another. In a previous paper[1], which is concerned with the case of meson decay, we have reported that (i) the infrared catastrophe occurring to the probability of the meson decay accompanied by real photon emission can be cancelled by taking into account another radiative correction to the decay probability due to virtual photon processes, and that (ii) the ultraviolet divergence involved in the latter correction can be concealed into the observed masses and coupling constant. In this note we shall give a brief account of the result for the second example, the case of beta disintegration of nucleon.

According to the Fermi's model, the beta disintegration of nucleon is caused by the interaction Hamiltonian of the form

$$V(x)=g\bar{\Phi}(x)\bar{\varphi}(x)\beta B\psi(x)\Psi(x)+\text{conj.},\qquad(1)$$

where $\Psi(x)$, $\Phi(x)$, $\psi(x)$ and $\varphi(x)$ denote the spinor wave functions for respectively proton, neutron, electron and neutrino, the barred ones their adjoints, g the coupling constant, and β and B Dirac's sedinions respectively for lepton and for nucleon (see Table).

As the corrective Hamiltonian which is responsible for the radiationless disintegration, we have obtained, after calculations similar to those performed in the former case,

$$\delta_0 V(x)=\delta_0 V(x)_{\text{ultra}}+\delta_0 V(x)_{\text{infra}}+\text{finite terms},\qquad(2)$$

with

$$\delta_0 V(x)_{\text{ultra}}=ga\bar{\Phi}(x)\bar{\varphi}(x)\beta\sigma_{\mu\nu}B\Sigma_{\mu\nu}\times\psi(x)\Psi(x)+\text{conj.},\qquad(3)$$

$$\delta_0 V(x)_{\text{infra}}=gb\bar{\Phi}(x)\bar{\varphi}(x)\beta B[2-p_\mu P_\mu F_0\times(p_\mu P_\mu)]\psi(x)\Psi(x)+\text{conj.},\qquad(4)$$

$$a=-\left(\frac{e^2}{4\pi}\right)\frac{1}{16\pi}\left(\log\frac{k_\infty+\sqrt{k^2_\infty+m^2}}{m}+\log\frac{k_\infty+\sqrt{k_\infty^2+M^2}}{M}\right),\qquad(5)$$

and

$$b=\left(\frac{e^2}{4\pi}\right)\frac{1}{4\pi}\left(\log\frac{m}{2k_0}+\log\frac{M}{2k_0}\right),\qquad(6)$$

where p_μ and P_μ represent the differential operators, $-i\partial/\partial x_\mu$'s, operating respectively on $\psi(x)$ and on $\Psi(x)$, m and M the masses of electron and of proton, k_0 and k_∞ the

Table

Type of coupling	scalar	vector	tensor	pseudo-vector	pseudo-scalar
g	g_s	g_v	g_t	g_{pv}	g_{ps}
β	1	γ_μ	$\sigma_{\mu\nu} = \frac{1}{2i}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)$	$i\gamma_5\gamma_\mu$	γ_5
B	1	Γ_μ	$\Sigma_{\mu\nu}$	$i\Gamma_5\Gamma_\mu$	Γ_5
βB	1	$\gamma_\mu\Gamma_\mu$	$\sigma_{\mu\nu}\Sigma_{\mu\nu}$	$-\gamma_5\gamma_\mu\Gamma_5\Gamma_\mu$	$\gamma_5\Gamma_5$
$\beta\sigma_{\mu\nu}B\Sigma_{\mu\nu}$	$\sigma_{\mu\nu}\Sigma_{\mu\nu}$	$-6(\gamma_\mu\Gamma_\mu + \gamma_5\gamma_\mu\Gamma_5\Gamma_\mu)$	$24(1+\gamma_5\Gamma_5) - 8\sigma_{\mu\nu}\Sigma_{\mu\nu}$	$6(\gamma_\mu\Gamma_\mu + \gamma_5\gamma_\mu\Gamma_5\Gamma_\mu)$	$\sigma_{\mu\nu}\Sigma_{\mu\nu}$

lower and the upper cut-off frequencies for virtual photons, $F_0(x)$ being the same function as that introduced in the previous paper. In deriving the above result the mass renormalization has already been performed.

$\delta_0 V(x)_{\mathrm{infra}}$ gives an infrared divergent contribution to the rate of radiationless disintegration of the order e^2, which is responsible for cancelling the infrared catastrophe of the radiative disintegration as expected from the general considerations on the infrared catastrophe[2].

As to the ultraviolet divergence, it is observed on comparing (1) and (3) that $\delta_0 V(x)_{\mathrm{ultra}}$, having a type of coupling different from that of the unperturbed Hamiltonian $V(x)$, can not be removed by the procedure of g renormalization, so far as a single type of coupling is assumed between nucleon and lepton fields, in contrast to the case of meson decay. If several types of coupling are assumed simultaneously, it is not impossible to remove the divergence with, in some special cases even without, the aid of g renormalization. Examples: (i) mixture of the scalar and the pseudo-scalar couplings, (ii) mixture of the vector and the pseudo vector couplings, and (iii) mixture of the scalar, the pseudo-scalar, and the tensor couplings. In the first or the second case, the condition $g_s = -g_{ps}$ or $g_v = g_{pv}$ suffices for the two divergent terms of the respective couplings to cancel each other without the aid of g renormalization. In the last case, the most simple g renormalization is $g_{ob} = (1-6\alpha)g$ under the assumption $g \equiv g_s = g_{ps} = 4g_t$ (see Table).

A fuller account will be published later.

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Errata: **On the Concept of the Nuclear Potential (5 (1950), 159)**

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In this letter there was a serious mistake, and the authors required its withdrawal after sending it, but it was not withdrawn due to an accident on business. The mistake was in several lines in the upper part of page 160. The deformation of the wave function due to the second order Møller interaction contributes a considerable part to the fourth effect, but is not all of it as stated there. The difference between the effect observed by Watson and Lepore and that due to the deformation constitutes the fourth order nuclear potential. The calculation of the higher order nuclear potential has also been carried out by Y. Nambu.